21. Exploring the Sketch Metaphor for Presenting Mathematics Using Boxer

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Abstract. The difference between sketches and paintings provides a metaphor for thinking about the structure of exposition in a computational medium. Drawing also upon Nevile's (unpublished) use of the term *literature* to indicate a particular stance in which ideas that have proved fruitful to someone at some time are made available to others in a dynamic computational medium, I explore the development of an example of such literature concerning modular arithmetic, presented in Boxer.

21.1 Introduction

What does it mean to "teach mathematics"? The metaphor explored in this paper is that of mathematics as a language of expression, like English or any other natural language. Early exposure is through enculturation, but refinement and expert control are enhanced through explicit teaching. There is a craft of expressing oneself, a literature of expressions by previous writers, and the possibility of improving one and analysing-criticising the other. Craft and literature interact to broaden the students' awareness of their natural language. To learn to use any language requires a need to express oneself, so there is an implicit assumption that students are exploring mathematical ideas and experiencing mathematical thinking in domains in which they have something to say or to sort out.

Mathematics teaching is currently dominated by training students in techniques, supplemented sometimes by problem-solving. Even the notion of problem-solving varies greatly, from rehearsing techniques on typical "questions" (the most common meaning) to tackling fresh contexts and exploring new avenues, variations, and generalisations (less frequently encountered). It is as if we spend most of our time as teachers training students in the reproduction of certain sentences typical of Shakespeare or Emerson. Teachers and text-authors often think they are teaching students mathematical analogues of sentence and paragraph structures (forms of argument, tricks and conventions) typical of mathematics, yet students are caught in particularities. They struggle to reproduce mathematical analogues of specific sentences (formulae and techniques).

Rhetoric about educational needs for the twenty-first century makes much of the need for flexibility. Students will need to deal with radical changes in working practices and thinking perspectives. Yet when expertise is transformed into instruction, an expert's awareness disappears and is replaced by trained student behaviour. Creativity and insight is turned into step-by-step instruction. One has only to contemplate the contrast between the potential of Logo (Papert, 1980), and the myriad of Logo worksheets and pre-prepared procedures that have been developed by fascinated teachers eager to "make it easy" for their students to encounter powerful ideas. Chevellard (1985) called this the *transposition didactique*. Kang and Kilpatrick (1992) related it to Buber's notions of I-You and I-It relationships in which an I-You relationship of an expert's lived and integrated experience is transformed into an I-It relationship of a student's rehearsed technique. What is taught is completely transformed from what was intended.

Exposure to good literature as conceived at the time has always been a backbone for natural language instruction, and in mathematics there are those who recommend exposing students to seminal papers, for example, Arcavi and Bruckheimer (1991), Barbin *et al.* (1988), Laubenbacher and Pengelley (1992), among many others. (For a bibliographic survey, see Fauvel, 1991.) In some cases the translation of a few fragments may be accessible to students, but on the whole, the styles of exposition and changing usage of technical terms in mathematics makes accessing original texts problematic in large doses. Yet exposure to literature provides examples of both style and content as well as perspective. It enables students to locate themselves as part of a continuing process, not just as passive recipients of established fact. Literature can provide stimulus for further exploration and expression by students examining implications of old expressions of ideas in a modern setting. All of this applies to mathematics as well as it does to ordinary literature.

Mathematical exposition can play a similar role, if it is distinguished from instruction. The point is not so much to learn what it says, but to explore how it says what it does, and even what it does not say; to investigate what happens when variations, alterations, and extensions are attempted. To understand something, it helps to locate boundaries, to find out what aspects can and cannot be extended. In language instruction, we have text processors which enable you to focus attention on ideas and their expression, and to worry about spelling and grammar later. In mathematics, we are only just beginning to see similar media emerge (diSessa, this volume). Computationally expressive media do not offer instruction by forcing repetition, but rather provide extensions of thought, enabling students to try out small changes and variations, to get to know the territory.

The notion of mathematical literature also opens the possibility of analysis and criticism: for example locating author intentions and influences, and relating these to the culture and mores of the time. Pimm (1988) offers examples of mathematical-literary criticism, showing how Fauvel (1988) is able to shed light on differences between Descartes and the Euclidean tradition through analysis of style; how Lakatos (1976) offers a fresh perspective on the relation between definitions and proofs; and how Leron (1983, 1985) employs computer science conventions to structure mathematical proofs in an approachable form.

21.2 Tools

Mustoe (1993) makes the point that even where software is designed specifically to support exploration, students in self-study mode (i.e., not in a laboratory with tutor and peer guidance, and stimulation) lapse into a passive mode. They tend to let the computer do the thinking for them. A true tool is an extension of the self: When you have a tool handy, you see problems through that tool, and you sometimes even reach for the tool before thinking deeply about the problem and whether that tool is most appropriate. Whereas in the past, software has only been able to check a particular case or example, symbolic and graphical manipulation packages are steadily increasing the level of generality and abstraction that they are able to manipulate. With increasing powers of abstraction, there are increasing challenges to work out how to use screens to contribute to teaching: deciding not to use a tool is as important as deciding to use it. Furthermore, the entrancing property of screens (Nevile, 1989; Mason, 1993), particularly electronic ones, attracting attention to the particular rather than to the general (Mason and Pimm, 1984), means that the challenges become ever more urgent for educators to address.

The design of tools to support mathematical thinking is intimately bound up with education of awareness in the use of those tools. Mathematics teachers and educators have for thousands of years struggled to find a suitable means of drawing student attention to important ideas, supporting them in making sense of those ideas, providing contexts in which to experience their use, and providing challenges on which to practice techniques to mastery. Each new technology has offered fresh possibilities to explore, and current developments are so rapid and on such a broad front that there will soon be fast graphics on every desk, perhaps in every palm, and access to the world's libraries at the click of a mouse, with corresponding implications for experts making mathematical ideas accessible to novices.

21.3 Teaching

How are students to be exposed to mathematics and the use of tools for expressing their mathematical thinking?

- Certainly not in some teacherly fashion, subject to the *transposition didactique*, in which expert awareness is turned into instruction in technique, and in the process, behaviour is substituted for awareness;
- Certainly not simply by providing a powerful computational medium, letting students loose on it, and hoping that they will encounter powerful ideas.

Any attempt to teach is caught by the didactic tension (Mason, 1986), in which

> the more clearly and explicitly the teacher indicates the behaviour sought, the easier it is for students to display behaviour without recourse to the understanding which is intended to generate that behaviour.

Neither pole is adequate as a refuge; something in between is sought, which liberates energy stored in the tension. Of course, every teacher has sought some ideal position on this spectrum, has prepared materials with an ideal format in mind, and with the image of students exploring and discovering, construing and learning. But once materials are in production, the multiple, nonlinearly-related imagined possibilities are turned into essentially linear actualities; much of the flexibility and dynamic is lost in the necessity to make certain choices. Material production is as much selection and editing as it is creating something new.

Rather than be caught in a simple tension between exploration and exposition, I try to locate a balance among six different modes of interaction, six different ways in which student, teacher and content can interact (Mason, 1979):

exposition	exploration	expression
explanation	examination	exercise

Exploration is currently being emphasised in mathematics education. But practice always lags behind ideals, and ideals tend to overstress the new at the expence of the old. In pushing for more student exploration as an integral part of learning mathematics, exposition has become undervalued. It is just as important as exploration, and the other ex's. In true exposition, an expositor uses the (possibly virtual) presence of an audience to make contact with a world of ideas, while in explanation the tutor enters the world of the student. My ideal form in which to encounter mathematical ideas involves a balance of all six modes, cross-referenced and interconnected. Boxer as an expressive computational medium may be able to facilitate rapid transitions from one interaction to another, making the learning experience more balanced and less fragmented.

The format I am struggling toward involves assertion of the theoretical or general in sufficient detail to provide a skeleton, drawing attention to key ideas. However, it leaves out enough detail (as in a sketch) so as to prompt the reader to specialise and explore in order to re-generalise with comprehension, and possibly to extend and vary for themself. The issue of level and amount of detail is perplexing, but I find it is enlightened by the metaphor of sketches and paintings.

21.4 Sketches and Paintings

Fish and Scribner (1990) draw attention to the importance of sketches rather than paintings as a metaphor for providing stimulus to students. A painting has richness of detail, but its completeness of detail means that the observer has to work in order to see through the whole, to make contact with and examine details and yet retain a sense of connection to the whole; a sketch provides just enough to invoke Gestalt powers of closure and to initiate a process of construal (Mason and Heal, 1993).

Mathematical sketches can consist of:

- diagrams and animations that appear to tell a story, and that induce a state of surprise, a need for explanatory or connective closure;
- statements of what is possible, bearing in mind that you don't actually need a manifestation in order to think about the problem. Indeed, screen manifestation attracts attention to the doing and away from the "thinking about";
- statements of what is usually true as a succinct mathematical outline;
- specialised procedures to enable the user to make relevant things happen, bearing in mind that most is learned in "teaching the machine" to do what you may or may not yet be able to do yourself.

What distinguishes exposition and sketch? The expositor enters a collector's state, wanting to integrate the apparently disparate and fragmentary, to map out the territory, to weave ideas into a coherent story, whether along historical or structural lines. There is a desire for completeness and for accessibility. This is the painter-in-oils. The expositor draws on the presence of the audience, both for preparation and presentation, in order to encounter the topic in a fresh way. During presentation, they hope to draw the audience into their world, rather like a tour guide who points out special sites and sights.

The sketcher works quickly to capture a mood, to indicate rather than summarise, to evoke desire for sense-making so that the viewer wants to reconstruct (imagined) detail. The viewer may then be moved to pursue and explore for themselves. There is a desire for speed and "tasting" rather than serving up a meal. The sketcher provides the audience with form, offering signposts to possibilities rather than route-maps. As Faux (1987) observed, the map is not the country, though the two can very often be confused.

21.5 Particular and General

The essence of mathematics is a constant movement back and forth between particular and general; what is general at one time often becomes particular later. Awakening students to:

- seeing the general in and through the particular
- the particular in the general

lies at the heart not just of mathematics teaching (Mason and Pimm, 1984), but of any discipline (Mason, 1984). Providing students with particular apparatus, particular software, particular diagrams and animations focuses attention on the particular. As noted in Mason and Pimm, electronic screens, particularly television screens, emphasise the particular, and it requires considerable effort to draw student attention to intended generality. Moving from words to images used to be thought important; now we can offer images more directly, but there is something very particular about an image on a physical screen that makes generalisation more rather than less difficult.

In the making of the Chinese Jigsaw domain with which I shall illustrate these ideas, my own attention was drawn to the screen objects, and away from general (theoretical) mathematical questions. To analyse the general I do not actually need to use a manifestation of a particular, but rather to find a notation that enables appropriate calculations to be carried out without actually doing them, *in absentia*, as it were. I am concerned that users, whose attention is naturally drawn to particularities of what is on the screen, may similarly not be supported to move into a more abstract notationally accessed world of general mathematical reasoning. The presence of tools to manipulate and extend what is on the screen attracts attention away from generality even with symbol processors.

By providing just enough detail for students to want to try to make sense by using their natural powers of specialising and generalising with entities that are confidence-inspiring for them, in a familiar and expressive computational medium, new life might be brought to mathematical literature. Symbol manipulation packages such as Maple, Derive, MathCAD, and Mathematica offer active worksheets in which students can invoke the symbol processor to redo calculations and try variations, can activate animations and change parameters. Geometrical manipulation packages such as Cabri Géomètre and Geometer's Sketchpad offer configurations that can be effectively infinitely varied to test for generality. Boxer offers considerably more as an embedding medium since the entire environment is active, responsive, and modifiable, and at no time is the user separated from the language which constitutes the software.

21.6 Task Design

An important feature of the design of tasks for students is to distinguish between what Tahta (1980, 1981) called *inner* and *outer* aspects of the task. The outer task is the behaviour, the "doing" described and sought. Such behaviour has both personal (psychological) aspects and collective (social) aspects. The inner task ranges from the purely personal (what opportunities are afforded for observing and working against personal propensities), the mathematical (what mathematical themes, processes, and awarenesses are experienced), the metamathematical (what more general forms of thinking, heuristics, and principles are exemplified), and psychological resonance (what aspects of social and psychological interaction are metaphorically present and able to be resonated, such as freedom and constraint, infinity and the unknown, multiplicity of viewpoints, etc.) (Mason, 1992).

Boxer provides a medium in which users can express themselves textually, diagramatically, and computationally. However, care must be taken to distinguish between boxes as containers for mathematical tasks (which will merely reproduce current text-based practice), and boxes as modifiable tools for exploration in and with mathematics. The distinction seems clear in theory, but it blurs in practice with software, just as it does with apparatus, workcards, and investigations.

Workcards are intended to stimulate exploration, but usually, because of the working of the implicit didactic contract and didactic tension, they evoke minimal behaviour needed to complete the task. Inner tasks are thus avoided or circumvented. Resource boxes in which something happens and in which structures are provided to make actions available (but which can also be extended and developed as desired) are, in my experience, much harder to construct. The *transposition didactique* arises whenever an expert attempts to instruct others in what they know rather than indicating some of the attractions of where they have been.

Many years ago I built a text-based example featuring the idea of winding numbers and rolling circles around different shapes (Mason, 1987). The basic format was a collection of related questions for investigation with some background information (including historical remarks) referenced to reflective comments on aspects of mathematical thinking. I now see this as the basis for a new form of mathematical literature in a computationally expressive medium. Boxer offers real opportunities for making a step forward by being able to integrate comments and references with interactive exploration.

21.7 Chinese Jigsaws: An Exemplary Domain

The dimensions being explored are:

- sketch vs painting;
- offering generality as fodder for specialising and direction for re-generalising;
- forms of commentary.

One place to start is with the fairly general:

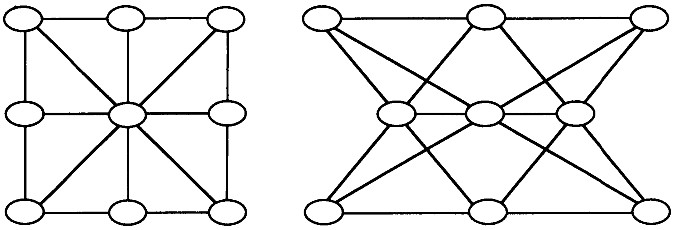


Figure 1. Chinese Jigsaws. A number of objects are provided, at the vertices of some configuration: Each lozenge represents an object, and to each line there corresponds a specified symmetry of each of the objects on it. A move consists of choosing a line, and then acting each of those symmetries on the corresponding object. The task is to return all the objects to some standard position.

This is far too general to be tractable, perhaps even to make much sense of. To make a start, perhaps guided by memory of traditional puzzles, lozenge objects can be replaced by coins or cups, which have two states and the two-dimensional configuration can be reduced to one dimension:

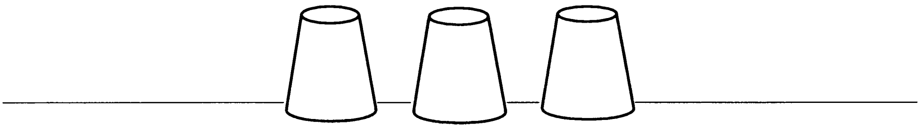


Figure 2. Flipping Cups. Three cups are in a line, all upside-down. The challenge is to get all three the right way up by flipping two-cups at a time.

This well known puzzle (Anderson, 1990) can be considered and easily dismissed. It is often given to pupils, and it even appears on cereal packages. When people tackle it, they usually become frustrated at not being able to succeed at what appears at first to be so simple and then they often stop work. If they attend to the patterns they *can* achieve, they can develop a sense of why they are not succeeding. An algebraic approach might observe that the parity of the cups is never altered by the moves, or might label each of the three basic moves (e.g., L for flipping the left most two cups, R for the rightmost, and O for the outside two), and then discover that there are connections between compound movements (e.g., $L^2 = R^2 = O^2 =$ *do-nothing*, $LR = O$, $RO = L$, and $OL = R$) which is the structure of the Klein Group. More importantly, any sequence of moves is reducible to just L, R, or O, or *do-nothing*, and since none of these succeed, the original task is impossible.

The puzzle can also serve as the starting point for development in a number of different mathematical directions. Yet how does one indicate these directions without making an oil painting?

The classic Logo philosophy (Papert, 1980) is to instruct the computer to do things, but without being caught up in problems of syntax and overhead of language-dominated thinking. In Boxer, one manifestation of the puzzle both enables the user to experiment immediately, and to extend and vary the conditions.

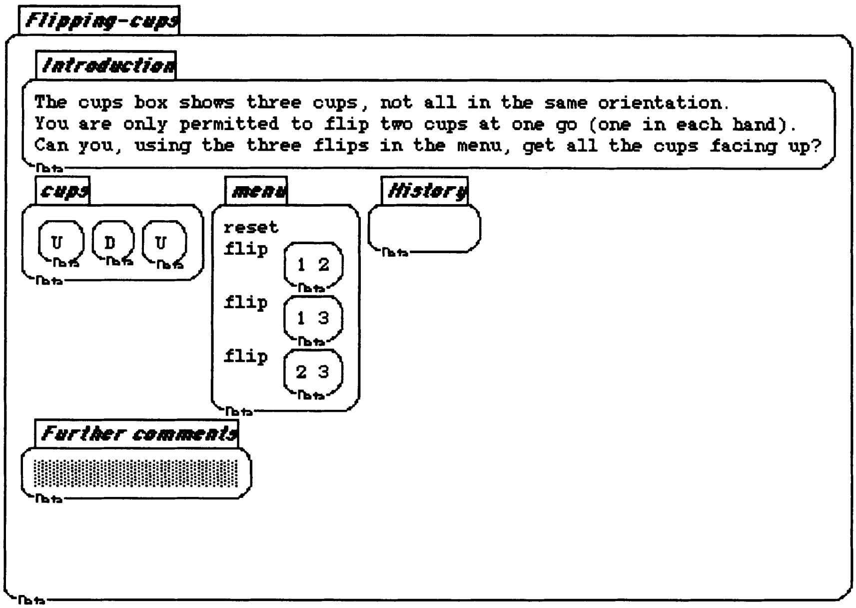
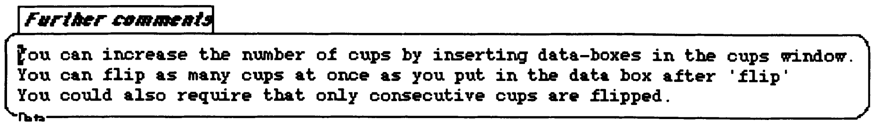


Figure 3. The Flipping Cups game in Boxer.

Note that the puzzle is essentially the same as the one posed above (Figure 2), but attention has been drawn to a central out-of-place cup as a common format to the puzzles that follow. The instructor in me proceeds by explicitly suggesting:



In this manifestation, the user can readily add more cups, and use more hands (i.e., flip more cups at one time), or restrict themselves to certain combinations of cups. It is not that the software is programmed to deal with a more general situation, but rather the tools provided make such extensions trivial. Of course the student would have to be sufficiently familiar with Boxer to recognise these possibilities. They can also add their own comments, and make whatever developments their familiarity with Boxer will permit.

Boxer makes it easy to provide ports from anywhere to a box of comments which can be accessed as a whole, or in part. Furthermore, users can add their own comments as an aid to themselves and to others, either at specific locations, or in a common comments box.

I would like to find some better way to indicate possible directions of development other than preparing them in advance, or offering suggestions. Perhaps however, this is something which best arises in and from interaction among students, software, and teacher. The only difficulty with such an approach is the need to support teachers in learning to work with students in this way. Extensions, variations and explorations develop out of interaction, arising from teachers' awareness in sensitivity to the moment, rather than being determined in advance as they are with workcards and textbook schemes. The temptation to prepare for all possibilities, to provide the student with what the teacher or student may not think of, is exactly the force which produces the *transposition didactique* and ruins so many innovative teaching ideas when they are presented to teachers or to students.

There are other aspects of the flipping cups which could be altered. With the original general situation in mind, re-generalisation and variations are possible. What if the cups were in some two dimensional array (as in Figure 4)?

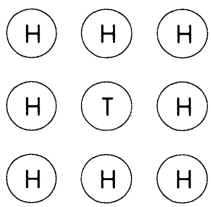


Figure 4. Flipping Coins. There are 9 coins in a three by three array. The centre coin shows tails, the others heads. You can flip all the coins in any row or column, and the aim is to get all the coins the same way up.

What if the cups had more than two states?

In place of cups, consider clock faces, where each 'flip' of the clock advances the hand one position (Figure 5).

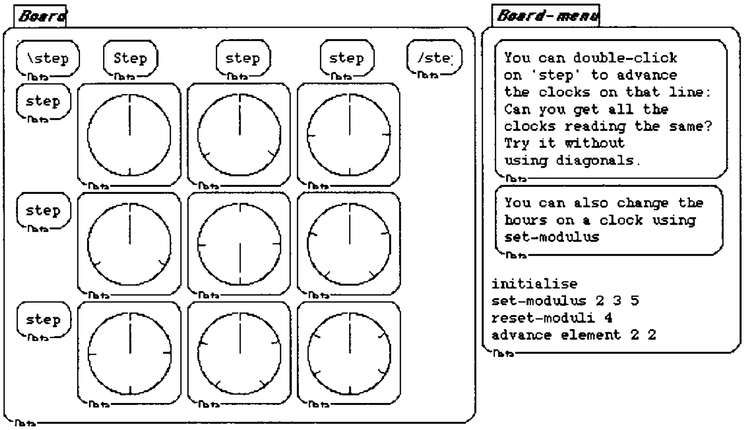


Figure 5. Clockfaces.

Here the *step* buttons activate the corresponding row or column, and the *\step* and */step* buttons activate the corresponding diagonals. Again this situation may be too complex to analyse immediately, so why not simplify matters by making all the clocks the same:

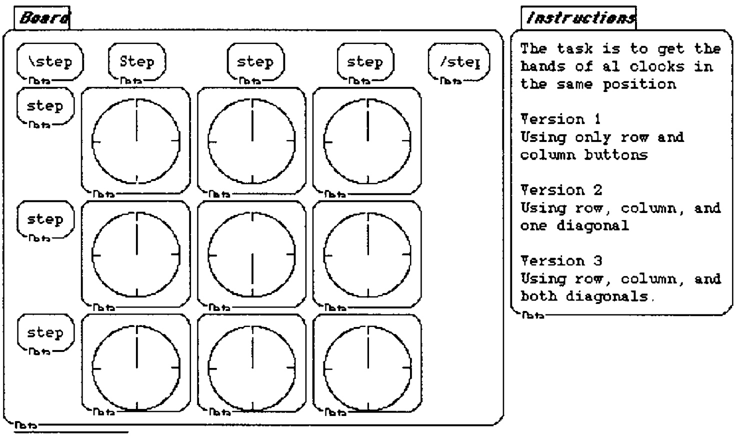


Figure 6.

The clocks could be replaced by objects with more complex (non-abelian) symmetries, which is in fact where the idea started. The origin of the name *Chinese Jigsaws* was a Chinese child's toy consisting of nine cubes that could be organised in a three-by-three array so as to display one of six different pictures. I discovered that once one picture was achieved, the others could be found by rotating each row, or each column about a corresponding axis.

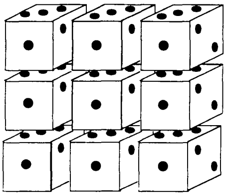


Figure 7. An array similar to the Chinese child's toy that was the origin of the Jigsaws puzzle.

Recently, I came across a reference in Surányi (1993) reporting that György Hajós posed a related question in 1969:

> A cube is placed on each square of a chessboard. The faces of the cubes are congruent to the squares of the board. Each of the cubes has at least one black face. We are allowed to rotate a row or column of cubes about its axis. Prove that by using these operations, we can always arrange the cubes so that the entire top side is black.

According to Surányi, its origin was one of Hajós' sons who liked to play with a picture cube puzzle like the Chinese one described above. An older brother teased him, spoiling the picture by rotating a row or column, and this inspired Hajós to pose his problem.

21.8 Aspects of a New Literature

Posing problems is an ancient and popular art. Providing sufficient sketch-like commentary to inform and guide without being oil-paintingly complete is not so common. The current version contains commentary linked via ports or locally situated comment boxes to specific points where difficulties may arise.

Sample Commentary referenced within the boxes:

> In mathematics, it is often the case that instead of showing how to do something, it is worthwhile to show that it is impossible. When exploring, it takes confidence in one's own intuition, one's sense of what is going on, to reach the conjecture that what you are trying to do is actually impossible.

When there are several operations available, it is useful to check whether the operations commute, because that can simplify the range of possibilities considerably.

When trying to analyse the effects of different moves, try to find some quality which is left unchanged by those moves.

When stepping back from work on a particular case and seeking a general approach to similar problems, it is often helpful to adopt a notation which reflects the important aspects of the operations, and suppresses the unimportant ones. For example, denoting by R_i the action of advancing each clock in row i by one position, and correspondingly by D_i and by C_i for the diagonals and columns, then sequences of these symbols can be used to denote the effect of them as operations.

It is rare to finish a mathematical exploration. Usually ones stops work, with at least some conjectures (together with supporting evidence), and preferably with some arguments to convince a sceptic that your findings are valid.

It is wise, when taking a break, to record current conjectures and any lines of investigation you have in mind, as they soon disappear from memory.

Sample Assertions referenced within the boxes:

Since there are 9 clocks but only 6, 7, or 8 operations, it is reasonable to expect that there must be some conditions on the clocks if they are all to be put into some pre-specified state.

The 3-by-3 array of clocks (all with the same modulus, and starting with all clocks showing the same 'time' except the middle one) can only be forced to have all hands the same if the middle hand shows a multiple of three. Even then it is necessary to use one diagonal.

Advancing all three rows advances all hands by one position; so if all the hands can be put into the same position, they can be put to any position; therefore we might as well assume they all return to the common starting position, which might as well be 0 or 'twelve o'clock.'

Since the row, column, and diagonal operations all commute, it is possible to write down equations for each clock that express the desire that they all change by 0 (except the centre one). Those equations show that the centre clock must start at a multiple of three. One consequence is that if the clocks have only three positions, then it will be impossible to get them all to read the same.

The purpose of these assertions is not necessarily to be clear, but rather to signal directions for further development. Assertions like these may, of course, intimidate rather than stimulate, but being in boxes that have to be opened to be read, there is less psychological weight attached, since they can literally be ignored.

Sample Suggestions for extensions and variations include:

How can the parity argument for flipping cups extend to more clocks?

For which numbers of cups c will h hands be able to convert a row of upside-down cups to all right-side up?

In each case, what proportion of the possible configuration-states are accessible from a given starting position?

What is the role of the centre clock in the 3-by-3 array?

Replacing clocks with dice and, more generally, objects with complex symmetries;

The rotating table problem[1] (Gardiner, 1979; Laaser and Ramshaw, 1981);

If different clocks have a different number of positions, under what conditions can they be aligned?

The more general problem with different numbers of hours in each clock is not fully resolved. Even in the case with identical clocks there are questions about how many different configurations of hands cannot be transformed into each other.

21.9 Reflections

The outer task in individual cases is to align some cups or clocks. Inner mathematical aspects include recognising that some tasks may be impossible, and generating arguments to prove impossibility through locating an invariant; using modular arithmetic; use of notation to encapsulate actions; use of sequences of symbols (words) for multiple actions; experiencing the ideas behind Z-modules, leading to group theory and ring theory. Meta-aspects include the role of imposing further or fewer constraints in order to appreciate what makes something possible or impossible; logical argument; freedom and constraint.

There remains an abiding issue. How does the teacher encourage students to use the tools to explore mathematically? It is compellingly attractive to become engrossed in what the teacher-author sees as particulars (particular cases of a general structure). Students do not naturally pause, stand back, and look for principles common to several examples. They need support to recognise examplehood (Mason

[1] You encounter a circular table with four symmetrically placed doors. When all the doors are closed, you may ask for any two to be opened. This reveals a tumbler in each, which may be either up or down. You may invert either, neither, or both tumblers. When the doors are closed, then if the four tumblers are all up, or are all down, a bell rings; otherwise the mechanism rotates so that you no longer know which tumbler is behind which door. Can you make the bell ring? Ahrens and Mason (unpublished) determined the number of hands needed as a function of the structure of the group acting on an arbitrary number of 'tumblers' with p possible positions.

and Davis, 1989), to appreciate that it is possible and desirable to look through the particular to a generality which is being exemplified.

How can students be supported in seeing the general through the particular rather than merely looking at the particular? My conclusion is that this process is most effectively encountered and adopted by being in the presence of others for whom it is natural, and who are sufficiently aware of their own awareness that they can choose to be explicit, indirect, or implicit in drawing attention to it, as seems appropriate in the situation. They need to be in the presence of experts manifesting important awarenesses such as seeking invariance and using notation. Thus teachers serve as role models for enculturation (Vygotsky, 1978), and act to scaffold-and-fade (Bruner, 1986; Brown *et al.,* 1989; Love and Mason, 1992).

Popper (1972) described the body of mathematics as stored in libraries as a *third world* of objectivity. There are many pointed questions to be asked about the objectivity of such a world. But in any case, it is currently a dead world, static and unwelcoming. By providing interactive mathematical literature in a computationally expressive environment, there are possibilities for making mathematical ideas a living reality for many more students than is currently the case. They could get access not only to ideas and techniques, but to different styles of presentation, and different styles of doing mathematics and of mathematical thinking. They could participate in a growing cultural heritage that would constitute a living, dynamic world of experience, like the one experienced by experts.

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