8. Microworlds as Representations

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> *It is as though, in becoming electronic, our beautiful old astrolabes, sextants, surveyor's compasses, observatories, orreries, slide rules, mechanical clocks, drawing instruments and formwork, maps and plans—physical things all, embodiments of the purest geometry, their sole work to make us at home in space—become environments themselves, the very framework of what they once only measured.*
>
> —M. Benedickt

Abstract. This paper examines in detail the category of open-ended exploratory computer environments which have been labeled "microworlds." One goal of the paper is to review the various ways in which the term "microworld" has been used within the mathematics and science education communities, and to analyze a number of examples of computer microworlds. Two definitions or ways of describing microworlds are proposed: a "structural" definition which focuses on design elements shared by the environments, and a "functional" definition which highlights commonalities in how students learn with microworlds. In the final section of the paper, the notion that computer microworlds, or symbol systems in general, can be said to "embody" mathematical or scientific ideas is examined within a broader consideration of ideas about representation.

8.1 Introduction

Mathematics and science have, over time, involved the use of various representational systems for expressing ideas and operations; and mathematicians, scientists and teachers have embodied these systems of representations in a variety of media, from figures drawn in the sand through pixels illuminated on visual display terminals. The purpose of this paper is to examine, in detail, what it means to say that mathematical ideas and operations are "embodied" in a particular kind of representation, and to focus specifically on a newly-emerging class of artifacts for learning mathematics, which are often labeled "microworlds." One of the first tasks of this paper will be to discuss just what is meant by the term "microworld," both by reviewing the various ways in which the term has been used by members of the mathematics education community, and by analyzing examples of particular microworlds. In examining these learning environments, we will look for commonalities in structural features or design elements, as well as in the way these systems are used by learners; that is, in what could be called the "functional" aspects of microworlds. Each of these views highlights different aspects of the notion of a microworld: In one case, the focus is on regularities in the artifacts themselves, on design elements shared by various examples of computer microworlds. In the other case, the focus is on how microworlds are used by individuals in learning situations, in which the environments can be seen to function as arenas for discovery, hypothesis-testing and learning.

The final section of the paper will address more closely the question of microworlds as representational systems. We will ask: What does it mean to "embody" some subdomain of mathematics in a computer microworld? How do students attribute meaning to what they see on the computer screen? And how are these meanings used by learners in a process of coming to understand a mathematical domain? In order to fully address these questions, we will draw from philosophy, psychology, and linguistics, as well as education, utilizing theories of meaning and representation, research on learning, and ideas related to the social construction and negotiation of meaning.

8.1.1 Preliminary Comments on Representations

In the process of coming to understand more deeply the activities of teaching and learning, a number of educators and researchers have turned their attention to systems of representations of mathematical and scientific ideas (e.g., diSessa, 1993; Goldin, 1988; Greeno, 1991; Kaput, 1987; Janvier, 1987). A potentially useful distinction can be drawn between "internal" representations, which are constructed by the learner and which may involve both conventional and "private" imagery (cf., Hadamard, 1945), and "external" representations: socially-shared, externally-displayed notations and means of expressing ideas which are encountered in the course of learning about mathematics or science.[1] Goldin (1991) distinguishes among four interpretations of the notion of representation in mathematics: "external physical embodiments (including computer environments)"; "linguistic embodiments"; "formal mathematical constructs"; and "internal representations," where the latter two concepts are characterized as "formal structural or mathematical analyses" and "students' internal, individual representations" (op. cit.).[2] These distinctions can

[1] Indeed, although socially-shared notations may be particularly well-specified in mathematical and scientific domains, shared external representations and vocabularies are constructed and negotiated by practitioners working within most other domains of human activity and thought, e.g., art, literature, engineering, and so on (Resnick, Levine and Teasley, 1991).

[2] This distinction corresponds to Tall, Vinner, and Dreyfus's distinction between concept definition and concept image, cf., Tall and Vinner, 1981; Vinner and Dreyfus, 1989.

serve as a useful starting point; however, one goal of this paper is to address the complexity of the notion of representation, looking specifically at how computer environments have been utilized to craft external representations of mathematical ideas. It is beyond the scope of this paper to address in any detail possible internal forms of representation or cognitive architectures; this is an area of much research and theoretical debate, both within and outside of mathematics education. However, the question of how external representations, whether conventional or newly-developed, can support or hinder learning in mathematics and science *will* be addressed, with the goal of better understanding the purposeful design and use of external representations in teaching and learning.

8.2 What are Computer Microworlds?

Computer microworlds can be looked at as specific forms of external representations. A question of vital interest to mathematics educators concerns the design and use of computer environments and other external systems of representation for teaching and learning about mathematical ideas. Goldin has phrased the question as: "How can we develop new external systems of representation that foster more effective learning and problem solving?" (1991, p. xxii). diSessa has summarized the same issue as follows:

> The program is, briefly stated, to transform old or invent new representations of physics, mathematics or whatever subject, which do justice to the powerful logical structure of the subject, but which at the same time mesh properly with the cognitive reality of human beings. (1979, p. 239)

The first section of this paper will address a relatively new medium available to those interested in supporting mathematics learning, that of the computer. Since the time that computers have become available for use in educational settings, many concerned with improving the teaching and learning of mathematics have looked to the power and flexibility of computational media to enhance their efforts. As a result, a wide range of computer environments for learning mathematics and other subjects have been developed and used in the classroom and research laboratory (for overviews and summaries, see Fey, 1989; Mehan, 1989; Pea and Sheingold, 1987; Wenger, 1987). Among the growing collection of software for mathematics education there exists a class of environments which have been called "microworlds" (or "discovery learning environments" (Shute and Glaser, 1990) or "intrinsic models" (Dugdale, 1991). Speaking loosely, microworlds have been described as computational environments which "embody" or "instantiate" some subdomain of mathematics or science. During this section of the paper, we will review various definitions of "microworld" utilized by researchers, developers and educators, tracing the evolution of the term in order to see whether a convergence on its meaning has been reached. In subsequent sections, we highlight structural and functional features shared by these environments, to illuminate what is characteristic about microworlds for learning in mathematics and science.

8.2.1 Early Uses of the Term "Microworld"

In his book, *Mindstorms,* published in 1980, Seymour Papert offers the following description of the idea of a microworld:

> ...the Turtle defines a self-contained world in which certain questions are relevant and others are not...this idea can be developed by constructing many such "microworlds," each with its own set of assumptions and constraints. Children get to know what it is like to explore the properties of a chosen microworld undisturbed by extraneous questions. In doing so they learn to transfer habits of exploration from their personal lives to the formal domain of scientific theory construction. (1980, p. 117)

This early definition has been extremely influential in the development of programs labeled "microworlds," and it may be useful to look closely at the original context in which Papert offers his description of this new class of exploratory environments (Hoyles, 1993 analyzes how the meaning of microworld has been transformed as its use moved from designers to schools). Papert describes microworlds initially in the context of the Logo "turtle," which can be either a mechanical device or a graphical object on a CRT screen that responds to commands typed in at the keyboard. A Logo programmer can, for example, give commands to the turtle to move forward or backward, or turn to the left or right, and can combine commands into more complex procedures. Abelson and diSessa, in their book *Turtle Geometry* (1980), explore in depth the distinctive kind of geometry which arises when these simple operations are made part of a programming language. In the quotation above, Papert notes that learners can investigate not only regularities in "turtle behavior" but also can explore Logo-based environments purposely built for exploration of other mathematical ideas. A wide range of Logo microworlds have been created in the years since Logo first surfaced; some of which will be discussed in the following section. Although Papert's use of the term "microworld" arose in the context of introducing the Logo programming language, and although the first "wave" of creators of microworlds worked in Logo, there have been microworlds and similar environments programmed in other computer languages. Current usage does not restrict the term to Logo-based learning environments (cf., McArthur and Lewis, 1991), or indeed, to computer environments at all (Rauenbusch and Bereiter, 1991).

8.2.2 Logo-based Microworlds

In the time that Logo has been available on microcomputers, many teachers and researchers have created Logo programs for learning mathematics (Charischak, 1989; Hillel, 1987; Hoyles and Noss, 1992; Leron and Krumholtz, 1989). In this section, I will not attempt a comprehensive review of these programs, but instead will focus on a few representative microworlds in order to clarify the characteristics of these open-ended, exploratory mathematical environments. I will also attempt to trace the various uses and the evolution of the term "microworld" within the Logo educational community.

The Dynaturtle

Before turning to microworlds which focus on mathematical topics, it is worth looking at an early microworld developed for exploration in physics, the "dynaturtle" (diSessa, 1982). The dynaturtle, developed by diSessa and later extended by White (1981), exemplifies the idea of a self-contained world in which scientific regularities or "laws" are implicit, waiting to be "discovered" by the learner. In this case, the regularities embodied are Newton's Laws of Motion, and learners are able to induce these laws by interacting with a turtle which is programmed to behave as an object in a frictionless universe. It is important to note that Newton's Laws are not stated explicitly or described verbally in any way in the microworld; instead, a computational object (the "dynaturtle") is constructed and programmed to behave so that it "obeys" these laws. This non-explicitness about the phenomenon under investigation (or concept to be learned) is one feature which distinguishes microworlds from other modes of instruction using computers; for instance, computer-based tutorials or other direct presentations of concepts and information.

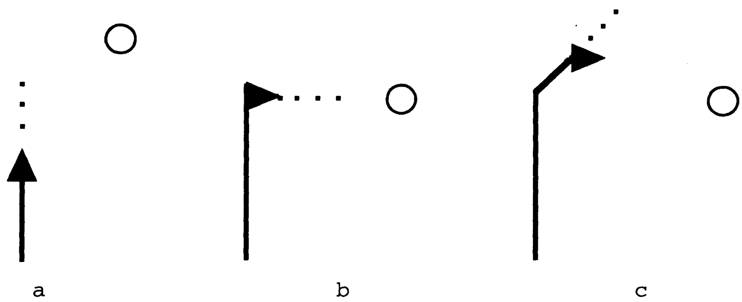

Figure 1. Dynaturtle (a) moving upward after a KICK; (b) expected result from a sideways KICK; (c) actual result (after diSessa, 1982).

diSessa describes the dynaturtle as "a graphics entity which can be moved around on a CRT with commands typed at a keyboard...a dynaturtle never changes position instantly, but can acquire a velocity with a KICK command which gives it an impulse in the direction the dynaturtle is currently facing" (1982, p. 39). In other words, the user can turn the turtle, and apply "kicks" to it; the turtle responds by continuing its forward motion unless a new force is applied to it, and furthermore, it "obeys" the parallelogram rule of vector addition for composing forces.

diSessa and White have used the dynaturtle for investigations of students' naive theories of motion, and the environment has been effective at eliciting students' expectations in this area of physics. Of interest to the current discussion is the structure of this early Logo microworld. Two notable features are, first, the linkage of a set of symbolic commands with graphical outputs specifically programmed to behave in accordance with scientific "laws," and, second, the provision of computer-based games which make use of these commands. In the case of the dynaturtle, one example of a game has the student apply kicks and turns in order to hit a target with a minimum speed. Another game has students "steer" the turtle around a corner. In both cases, an important aspect of students' learning is that they are able to compare their expectations about how the turtle will behave with the actual outcome, and by interpreting the feedback provided by the environment, clarify their understanding of how Newton's Laws actually work. It will be argued that this "learning dynamic" is a feature of many microworlds, and in fact, may be useful as a defining element for such environments.

Mathematical Microworlds in Logo

The Logo community flourished after Logo became available on microcomputers, and many mathematics educators responded to Logo's availability, its obvious connections with mathematics, and Papert's vision as expounded in *Mindstorms* by designing mathematical microworlds in Logo. One reading of Papert's original description of the Logo turtle would hold that Logo itself (or at least, those aspects concerned with turtle graphics) is a "microworld" for mathematics, or for a version of differential geometry. In this section, we will examine examples of microworlds in which Logo procedures are specialized to focus on other specific subdomains within mathematics.[3]

A detailed research program on the use of Logo in learning mathematics has been carried out at the University of London since the early 1980s, comprising projects working with both schoolchildren and teachers. The work of this group, whose major participants include Hoyles, Noss and Sutherland, will be used as a case for examining Logo microworlds for mathematics, and for looking at the evolution of the use of the term in this particular community.

In an early paper on the potential of Logo in the learning of mathematics, Hoyles echoed Papert's original formulation of the idea of microworlds by noting that "an almost endless variety of structures, which add some facilities and possibly curtail others, may be superimposed upon Logo in order to focus the learning experience around specific mathematical concepts" (1985, p. 32). The microworlds developed by the University of London group are based upon this model; their starting point is Logo, which is used to create specific procedures designed to assist the student to learn about certain mathematical concepts.

[3] Lawler (1987) has proposed a distinction between Logo "miniworlds" and "microworlds," in which miniworlds would comprise such general areas of explorations as list manipulation, procedure definition, various kinds of turtle geometry, and Lego/Logo devices. Examples of microworlds, in Lawler's framework, include "sentence generators, sprite based word worlds...geometric designs, target shooting games, child-designed arithmetic CAI, multiple representations for algorithms" (1987, p. 18). In the definition of microworld being developed in this paper, this mixed list of Logo activities would include some exemplars of microworlds and some non-exemplars.

One category of mathematical concepts for which this group created a set of microworlds was Ratio and Proportion. The basic pattern for all the Ratio and Proportion microworlds was the same: An initial procedure was created by the researcher which drew a graphical object on the screen (it might be a stick figure, a block letter N, a house, or a parallelogram). The learner was introduced to this procedure in Logo, and invited to modify it in some way (or to explore it by completing "guided discovery" worksheets). The microworlds and the activities were designed, as Hoyles and Noss state, "so that the pupil [would] bump into embedded mathematical ideas in the context of meaningful activity" (Hoyles and Noss, 1987a, p. 134). The work with the microworlds typically took place in pairs with fellow students, but was guided and monitored by the researchers and/or the teachers.

A brief excerpt from a description of the HOUSE microworld will give the flavor of these early explorations of Logo microworlds:

> Pupils were given a procedure for a closed shape—HOUSE...and the procedures STEP and JUMP which, respectively, moved the turtle (without drawing) up and across the screen. We made these procedures available in order to avoid pupils having to confront the problematic issues of interfacing procedures and turtle orientation....Pupils were asked to build bigger and smaller HOUSEs that were all in proportion. We hoped that pupils' attention would be drawn to the *necessity* of using multiplicative scalar relationships, since unclosed or overlapping shapes would be produced as a result of adopting nonmultiplicative strategies. Our model was thus based on the idea of informative and surprising feedback triggering cognitive conflict which would lead to a reevaluation of pupils' strategies. Figure [2] illustrates computer feedback from the adoption of an additive strategy and the obvious mismatch between the intended and actual outcomes. (emphasis in original, Noss and Hoyles, 1992, p. 446)

```
TO HOUSE
HT
FD 50
RT 60
FD 70
RT 60
FD 70
RT 60
FD 50
RT 90
FD 121
RT 90
END
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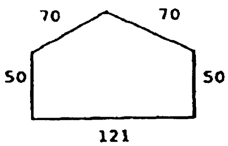


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TO BIGHOUSE
HT
FD 125
RT 60
FD 145
RT 60
FD 145
RT 60
FD 125
RT 90
FD 196
RT 90
END
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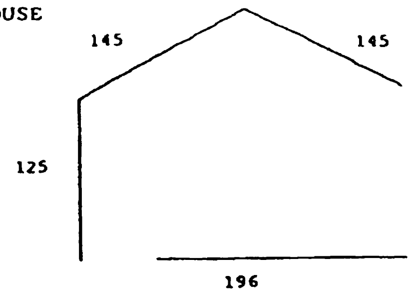

Figure 2. HOUSE, a ratio and proportion microworld in Logo (Noss and Hoyles, 1992).

In these and other microworlds, there is a presumption that the learner knows and will use the Logo programming language as part of his or her explorations. This presumption that Logo is a part of microworld explorations is reflected in other comments in this thread of work; for example, in discussing a syllabus for algebra, Noss states the possibility that one could "create a system of microworlds (or perhaps just one, providing it is rich enough) based on the ideas of algebra. In other words, algebraic concepts could be introduced via their symbolic representation within Logo programs" (1986, p. 354). It is not the purpose here to debate whether this approach to learning mathematics in a Logo context is valuable, but simply to note that, as will be discussed further in a later section, the idea of a microworld does not *necessarily* involve Logo programming, modification of Logo code or interaction with Logo.

In early work done in this vein, Hoyles and Noss, like others working with Logo, identify the set of Logo procedures focusing on a specific mathematical topic or subdomain as "the microworld." However, in a later definition, the Logo procedures which comprise the computational environment are considered as only one component of the microworld. Hoyles and Noss (1987) propose a definition or framing of the concept of a microworld which includes a technical component (the software/hardware aspects), a pedagogical component, a pupil component, and a contextual component, stating that, "A microworld cannot be defined in isolation from either the learner, the teacher or the setting; activity in the microworld will be shaped by the past experiences and intuitions of the learner, and by the aims and expectations of the teacher" (ibid., p. 587). This is an extremely important point, which will be taken up in more depth later in this paper. However, for clarity's sake, the usage of the term "microworld" in this paper will be restricted to what Hoyles and Noss refer to as the technical component, that is, the software or artifact itself, purposefully created for exploration of mathematical or scientific phenomena.

8.2.3 Refinements of a Definition and Further Examples

In an essay entitled "In Search of Piagetian Mathematics," Groen and Kieran discuss and elaborate upon Papert's original description of microworlds:

> ...*microworlds*...are essentially mini-domains of Piagetian mathematics. They can be formulated as mathematical systems with axioms and theorems (e.g., Abelson and diSessa, 1981), but these lie beneath the surface as far as the child's direct experience is concerned. The transformations manifest themselves as commands that result in changes in the states of concrete objects. The effects of these commands are governed by the axioms (which are frequently simply descriptions of the outcomes of commands). The theorems manifest themselves as general properties of combinations of commands. (1983)

In this passage, Groen and Kieran point to what is probably the most obvious, but fundamental, aspect of a computational microworld: that the scientific or mathematical phenomenon which the designer intends to introduce to the learner is instantiated or embodied in computer code. It is by translating mathematical or scientific regularities into procedures and computational objects that the designer constructs a microworld, and this process involves a complex series of choices and design decisions (cf., Edwards, in press; Pirolli and Greeno, 1988; Pratt, 1991). Note that the description of a microworld given by Groen and Kieran does not imply that the learner him or herself is involved in programming, as is assumed in many of the Logo-based microworlds. Instead, microworlds can be created where both the underlying code, and the underlying mathematics embodied in the code are, at least initially, not transparent to the learner. In this case, the designer creates a world "on top of" the original computer language, and the learner must then construct an understanding of this structured world.

At this point, I would like to note that when a particular mathematical or scientific subdomain is "deconstructed" in order to be reconstructed in computer code, the designer is carrying out a sophisticated analysis of the domain, and is actually involved in building a computational (or mathematical) model of the phenomenon or system in question. These models, although constructed with instructional goals in mind, are analogous to computational models created for the purpose of scientific research; for example, models of chemical structure or of weather systems. The main difference may be that when researchers construct mathematical or computational models of natural phenomena, these models are often "incomplete," in the sense that empirical data is used iteratively to refine the models. In educational microworlds, we generally choose domains for which we are able to build a fairly complete and stable model for the learner to explore. These microworlds or models may very well provide access to ideas and phenomena which are not otherwise easily encountered by students. For example, part of the power of the Dynaturtle microworld is that it provides an arena in which students can vicariously "experience" a frictionless universe, and though experimentation and feedback, build a set of intuitions about this domain. Wilensky has constructed a microworld, using a massively parallel version of Logo called *Logo, in which users can experiment with the behavior of thousands of gas molecules; he refers to his world as "an environment for playing with and exploring large ensemble behavior" (1993). A microworld for hypercubes and other higher dimensional mathematical objects can be imagined, again, providing students with experiences which are very difficult to present using more static forms of media.

The power, in terms of learning opportunities, of having the students *themselves* carry out the analysis of a mathematical or scientific subdomain, and instantiate the results in computer programs, is a feature of both early and recent uses of Logo and related environments, but one which will not be explored in detail in this paper (see, however, Adams and diSessa, 1991; diSessa, 1989; diSessa, Eisenberg, this volume; Harel, 1991; and Sherin, diSessa and Hammer, 1993 for recent discussions and examples of this notion).[4]

Groen and Kieran discuss another aspect of microworlds, in pointing out (as do Papert and others) that microworlds do not need to be based on a computer. They state, "Microworlds can exist outside of the computer. It could be said, for example, that informal arithmetic based on counting is a microworld" (1983). A fuller discussion of non-computer microworlds will be reserved for a later section, but in their essay, the authors also point out the power which a computer embodiment lends to a mathematical domain, the capability for what they call "self-correction"

> ...the pupil may invent a grossly incorrect or "buggy" theory about the microworld. In a non-computer setting, it may be difficult for the student to become aware that something is wrong....In contrast, a computer-based microworld is naturally self-correcting. If a program does not execute as anticipated, it is clear immediately that something is wrong. The nature of the errors may yield additional information. If the cause is nontrivial, the task of debugging or discovering the cause of the error may lead to major modifications in the theory. (1983, p. 372)

Although it may be preferable to refer to students *using* the microworld in a process of self-correction (rather than describing the microworld itself as "self-correcting"), the point is vital: Computer microworlds provide feedback which allows the students to apply a process which has been called "conceptual debugging" to their existing theories about how the environment functions (Edwards, 1990). Hoyles and Noss have remarked upon this central feature of Logo-based microworlds, in which "the learner is both engaged in the construction of executable symbolic representations and is provided with informative feedback" (1987a, p.133).

Patrick Thompson has carried out an extensive program of research and development of microworlds which can be seen as "mini-domains" of mathematics, and has contributed a concise definition of a mathematical microworld:

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[4]An important issue, but one which will not be pursued in the current paper, concerns a category of mathematical exploration tools, for example, the Geometer's Sketchpad or Cabri, in which the graphical representations for mathematical or geometric objects are linked *not* to a separate set of commands which can be entered by the user (and combined, for example, into more complex sequences), but instead to items from a pull-down menu. It is an area for further empirical investigation, as well as theoretical analysis, to consider the differences in learning with exploratory environments which provide a separate "language" for mathematical entities compared to those which rely on the actions of selecting from alternatives presented on a menu. It may be that a set of word-like commands adds expressiveness and enhances the feedback between multiple modes of representation (aside from allowing for construction of complex sequences of operations, which is addressed in the geometry environments mentioned above by the provision of "scripting" or "macro" facilities). These and related issues are beginning to be addressed, for example, see Hoyles, Sendov, this volume.

> I will use "mathematical microworld" to mean a system composed of objects and relationships among objects, and operations that transform objects and relationships. This characterization is meant to capture the idea of a mathematical system as constructed from primitive terms and propositions, where the full system initially exists only potentially but includes features that allow students to expand that potential...In practice a mathematical microworld incorporates a graphical display that depicts a visualization of the microworld's initial objects. The display in conjunction with operations upon the microworld's objects constitutes a model of the concept or concepts being proposed to the students. (1987, p. 85).

Examples of Thompson's mathematical microworlds include INTEGERS, in which a turtle moves along a number line in the positive and negative directions; MOTIONS, a microworld for geometric transformations; and the BLOCKS Microworld, illustrated in Figure 3. In describing the latter environment, Thompson highlights another distinctive feature of microworlds, the dynamic connection of multiple representations: "I created a mouse-driven, computerized microworld, called *Blocks Microworld*, that presents base-ten blocks and decimal numeration as linked systems....Any change in blocks causes a change in the numeral; any change in a numeral causes a change in blocks" (1992, p. 127).

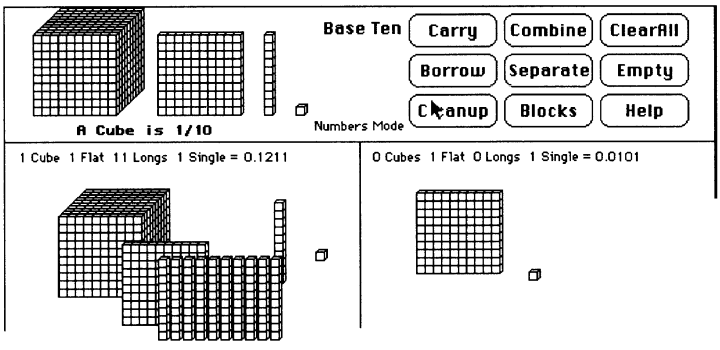

Figure 3. Screen display of BLOCKS microworld after a student has selected A cube is 1/10 in the Unit menu. (Thompson, 1992).

The BLOCKS microworld was designed to assist students to create a link in their minds between actions on numerals (a symbolic system) and actions with a computer representation of concrete objects. This emphasis on the linkage of more than one representation is another characteristic feature of microworlds, and is evident in a further example of a microworld, created by the author, called TGEO.[5] TGEO,

[5]Thompson and Edwards independently developed Logo-based microworlds for transformation geometry (as have other mathematics educators). The microworlds differ in certain choices made in representing transformations of the plane; for example, in the TGEO microworld, rotations are global mappings of the plane around any arbitrary center point, while in MOTIONS, rotations always occur around the origin. The effects on learning transformation geometry of this and other design decisions has not been investigated.

illustrated in Figure 4, links a set of commands for Euclidean transformations of the plane with a graphical window which displays the results of the transformations by showing the mapping of a block-letter L. Students can enter the commands to see the results of simple transformations of the plane (for example, `ROTATE 10 10 90` results in a clockwise rotation of 90° around a center point at (`10,10`)).

As with the microworlds developed by Hoyles and Noss, diSessa, Thompson, and others, TGEO has been used both as a research tool for investigating students' understandings within a particular domain, and as an instructional artifact. When used as an instructional tool, a curriculum of activities must be developed in which the students are provided problems, challenges, or interesting areas to explore. The core microworld itself is neutral; in contrast to computer-assisted learning environments such as drill-and-practice programs, there is not necessarily a built-in, predetermined sequence of problems or exercises in a microworld.

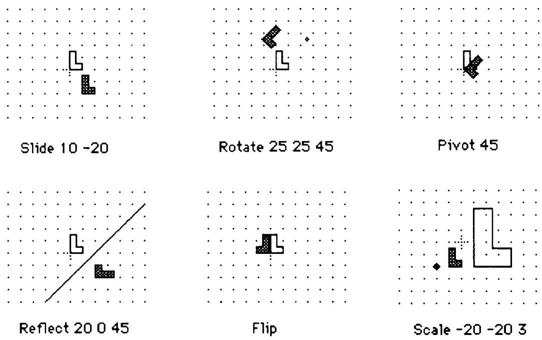

Figure 4. The TGEO microworld.

The microworlds illustrated in this section differ from those described in the previous section in that there is no presupposition that the learner must know or use Logo. Instead, each of these environments presents a self-contained mathematical world or mini-domain, with its own set of commands and operations, and in fact, could be written in any sufficiently-flexible computer language.

8.2.4 "Microworlds" By Any Other Name: Simulations, Intrinsic Models, Interactive Illustrations, and Discovery-based Learning Environments

In previous sections, we examined definitions for mathematical microworlds, and looked at several specific examples of environments which the designers have labeled "microworlds." In this section, we will consider environments which are quite similar to microworlds, but for which the designers have chosen different labels. The choice of a specific label for a learning environment (or the avoidance of the term "microworld") may be intentional; the purpose of this section is not to dispute specific labels, but simply to highlight that the characteristics shared by environments labeled as "microworlds" are also found in other interactive computer environments. This section will describe several of these related environments.

Microworlds and Simulations

Simulations and microworlds are closely related learning environments, as evidenced by a definition of microworld offered by Pea:

> A microworld is a structured environment that allows the learner to explore and manipulate a rule-governed universe, subject to specific assumptions and constraints, that serves as an analogical representation of some aspects of the natural world (1987, p. 137).

This definition, as do the previous ones cited, highlights the "rule-governed" aspect of a microworld, but goes on to state that a microworld represents "some aspects of the natural world." Although I don't consider the distinction essential, there are those (cf., Papert, 1987) who would separate "microworlds" from "simulations," reserving the latter term for environments which represent elements of the "natural" world. Pursuing this distinction would lead to the philosophical issue of whether mathematics "belongs" to the natural world, a topic which will not be pursued at this point (an excellent discussion of the issue is presented in White, 1956). However, if we agree to use the term "microworld" in Pea's sense, not excluding environments which model aspects of the natural world, it may help us to see more clearly characteristics which make such environments effective. As an additional example of a "microworld" or "simulation," one which makes use of linked multiple representations, we can examine Trowbridge's "Graphs and Tracks" (1989).

Graphs and Tracks: A Simulation/Microworld for Motion

Graphs and Tracks presents the learner with a set of linked representations, all of which involve motion, velocity and acceleration. As shown in Figure 5, the primary real-world object modeled in the environment is a set of ramps and supports, which can be directly-manipulated (Hutchins, Hollan and Norman, 1986) in order to set up a track along which a ball can roll. The initial velocity and location of the ball can be set by the user, who can also choose to display graphs showing the change in the ball's velocity and/or acceleration as it moves along the track. The user can freely explore the relationship between the simulated motion of the ball and the graphs representing this motion; furthermore, a series of challenges are available. In these challenges, the learner must try to create a track, as well as initial settings for the ball's position and velocity, which results in motion that "matches" prestored graphs. Through trial-and-error, or by reasoning through the relationship, the learner can eventually succeed at this task and, presumably, increase his or her understanding of the underlying physics.

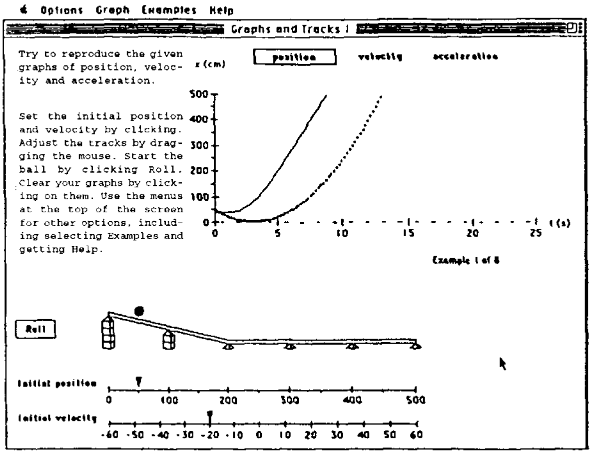

Figure 5. Graphs and Tracks (Trowbridge, 1989).

The essence of a microworld, as expressed in the various definitions reviewed so far, centers on the creation of a rule-governed environment made accessible for manipulation and exploration by the learner. Most of the definitions do not specify the nature of this exploration and manipulation, and certainly the specific task and activities appropriate for a given microworld would depend on the subject domain and on the objects and operations available in the environment. However, the task of "matching" the action of the rolling ball to its graphical representation is similar (when viewed at a sufficient level of generality) to puzzles, games or challenges embedded in other microworlds. We have seen that in the Dynaturtle microworld, the learner must try to reach a target (thus, at least implicitly, "matching" an effective sequence of kicks and turns). In the TGEO microworld, one of the central activities asks the learner to apply a sequence of transformations in order to superimpose two congruent shapes; this activity is called the "Match Game." I would like to offer a final example of this type of activity, drawn from a computer environment which was not labeled a microworld by its designers, but which shares many of the characteristics of one.

Green Globs: An "Intrinsic Model" for Functions

In the Green Globs game, created by Dugdale and Kibbey (1990), the student is presented with a linked representation of function, where one representation is the symbolic form of an algebraic equation, and the second is that equation's graph. Graph-plotters and similar tools are becoming commonly available on computers and calculators; what sets apart Green Globs are the learning activities built around this basic facility. The central activity is a game in which a set of points or globs is randomly scattered around the coordinate plane, and the learner must enter equations which pass through as many points as possible on a single round (Figure 6).

Again, as with Graphs and Tracks and TGEO (as well as other similar microworlds and computer-based learning environments), a central activity in Green Globs involves a game or challenge, whose purpose is clear to the learner. However, in order to succeed at the game, the learner must induce an understanding of how the mathematics or science functions; this is the underlying pedagogical goal of the activity (a further discussion of the role of this kind of activity in microworlds can be found in Edwards, 1992b and Dugdale and Kibbey, 1990).

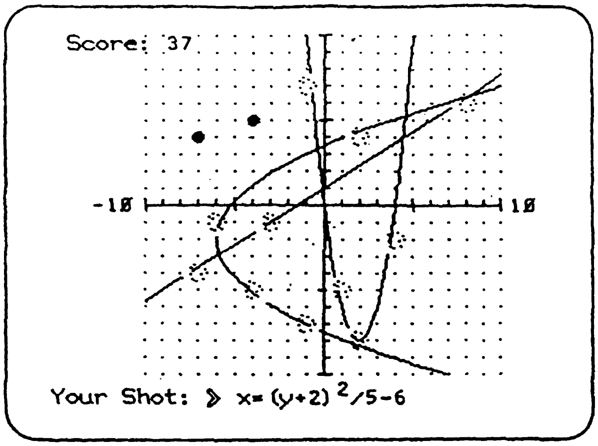

Figure 6. Green Globs (Dugdale, 1990).

Dugdale did not label her environment a "microworld"; instead, she uses the term "intrinsic environment," and presents the following list of characteristics of such environments:

1. Students are given a working model to explore and manipulate.
2. The mathematics to be learned is intrinsic to the model. In other words, the model is a direct expression of the underlying mathematics.
3. Feedback is direct, relevant, diagnostic, and often graphic, so that students can tell at a glance what the error was and how it relates to a correct solution. Unnecessary verbiage and gimmicks unrelated to the mathematics are avoided.
4. The model provides a rich environment for exploration by students of widely varying mathematical background and ability, but students find that the more mathematics they apply, the more they are able to do. (Dugdale and Kibbey, 1990, p. 203).

We can see how closely this list of characteristics fits the many of the environments labeled microworlds which we have looked at; indeed, I would characterize Green Globs as a microworld for functions.

"Interactive Illustrations" and "Discovery Worlds"

To conclude this section, I will mention in brief two other environments which are similar to microworlds, but which have been called by different names. Although their designers do not call these environments microworlds, they could be so labeled, as they share both structural features with microworlds and also function in the same way as learning environments. One example is a computer-based learning environment for fractions, developed by Ohlsson (1987). In this environment, Ohlsson has developed a graphical representation for fractions which makes use of rectangles in which the "height" represents the numerator, and the "width" represents the denominator. Ohlsson describes these representations as "interactive illustrations", which he defines as "graphical displays which are responsive to the learner" (p. 308). This "responsiveness to the learner" is a feature which Ohlsson's "interactive illustrations" share with environments labeled microworlds, as well as the fact that his interactive illustrations link a symbolic notation for fractions with a visual representation, within the context of a set of activities and problems to solve.

Shute and Glaser describe a microeconomics environment which is called *Smithtown*, which they describe as an "intelligent discovery world" or a "guided discovery environment." Again, they highlight the student-directed nature of such learning environments, stating:

> [T]here is no fixed curriculum. Rather the student generates his or her own hypotheses and problems, not the system. After generating a hypothesis, (e.g., Does increasing the price of coffee affect the demand for Cremora?'), the student tests it by executing a series of actions, such as collecting baseline data...This series of actions for creating and executing a given "experiment" defines a student solution. (1990)

As we will see in the next section, this kind of experimentation is a feature of microworlds and discovery environments, one which I will propose should form an element of a functionally-based definition.

8.3 What Makes a Microworld a Microworld? Two Views

In seeking a definition for an artifact we choose to label "microworld", there are two possible approaches. One approach is to attempt to specify a categorical list of characteristics or design features, ones which we can state will be found in every specific instance of the more general category "microworld." Although this approach has not always been effective (cf., Lakoff, 1988), an attempt to do so for microworlds will be made in the next section.

Subsequently, we will look again at a definition of microworld, but one which is based on characteristic learner interactions with such environments, on how this type of learning environment is *used,* rather than on a finite list of structures or features. This constitutes a functional approach to analyzing microworlds and related interactive learning environment (the "structural" and "functional" distinction in the design of computer environments is discussed in diSessa, 1986).

8.3.1 The Structural View

If we examine a computer microworld from the point of view of a designer, or if we review the examples of microworlds presented so far, we can distinguish a list of characteristics which seem to be common to all of these environments. This list can be used to create a "structural" definition of a microworld. Such a list or definition might include the following elements:

- A microworld contains a set of computational objects (defined formally through procedures or programs) which have been created to reflect the structure of mathematical or scientific entities within some subdomain of mathematics or science (or, alternatively, if we wish to include a wider variety of environments in this definition, which model some aspects of the natural or social world).
- A microworld links more than one representation of the underlying mathematical or scientific entities or objects. Typically, these representations include a symbolic and a visual or graphical component, although it is an area for further exploration and investigation as to what other modalities might be usable in the design of microworld-like environments (sound, motion, etc.).
- Often the objects and operations in a microworld can be combined to form more complex objects or operation; this is particularly so when one of the representations consists of a "language" for the entities and operations.
- Typically, a microworld includes a set of activities, which may be pre-programmed into the environment or which may be instantiated in worksheets or verbal instructions, in which the user is challenged to use the entities and operations to reach a goal, solve a problem, duplicate a situation or pattern, and so forth.

These seem to be the minimum structural features of a computer microworld. In the next section we will turn to some of the functional aspects of these environments; that is, to the features which become salient when we consider how microworlds are actually *used.*

8.3.2 The Functional View

It may be useful to separate the features built into a computer microworld by its designer from aspects of the microworld which emerge when it is placed in front of a learner; that is, from characteristics that become apparent only when the microworld is actually used. In previous definitions or characterizations of microworlds, including one offered by the author ("an environment, based in a computer or another medium, in which the central objects and relations of a domain are instantiated into a concrete or semiconcrete form that is accessible to new learners," Edwards, 1991, p. 123), the functional aspects of microworlds are often confounded with those aspects which form part of the design of the artifact.

For the kind of microworlds we have considered here, the functional aspects concern characteristic or typical ways of interacting with the environment which we see in students using the environment. These ways include the following:

In the use of a microworld, the learner is expected to:

- manipulate the objects and execute the operations instantiated in the microworld, with the purpose of inducing or discovering their properties and the functioning of the system as a whole. Experimentation, hypothesis generation and testing, and open-ended exploration are encouraged;
- interpret feedback from these manipulations (feedback which may be provided through multiple, linked representations) in order to self-correct or "debug" his or her understanding of the domain;
- use the objects and operations in the microworld either to create new entities or to solve specific problems or challenges (or both).

These functional features of microworlds-in-use, as well as the fact that microworlds typically have a broad curriculum of activities built around the core environment, mean that these environments are useful for students with a wide range of previous knowledge and experience in the domain. It is of course a matter of careful, theoretically-founded design to select or create representations which are appropriate for both the subject domain and for the target learners. Furthermore, once an initial design has been implemented, this is just the first step, to be followed by empirical testing of the design choices with the intended users. As diSessa has stressed, the design of interactive, exploratory computer learning environments best takes place within this kind of iterative cycle, in which an initial design is used with students and empirical information on their interactions and understandings is used to "feed back" into an improved version of the computer environment (diSessa, 1985; cf., also, Edwards, in press).

In analyzing student interactions with a microworld, it is important, although not always easy, to try to distinguish difficulties which arise from particular interface choices (and which can be ameliorated by an improved design) from those which are more deeply conceptual, and would arise regardless of the specific choices of symbol system or representation. These latter types of difficulties cannot be overcome through clever interface design, but rather are themselves evidence that the student is at the edge of a real learning opportunity, at a point where he or she must go beyond what is familiar and known. It is, in fact, in order to surface and challenge students' expectations about a new domain that microworlds are created—if the learners fully understood the nature of the mathematical or scientific phenomena they encounter in a microworld, the program would function perhaps as a practice environment but not as an exploratory, *learning* environment. It is during moments of surprise, when the unexpected happens, that the power of a microworld is most apparent. These are moments when the microworld does not behave in the way the user currently expects it to, when a student might exclaim, "Hey, this program isn't working." At these moments, the program is actually working optimally as a learning environment, because it is providing the experiential material which can allow a learner to reconstruct or "debug" his or her understanding of the phenomenon in question. This experimentation-feedback cycle is a hallmark of a microworld, when viewed from the functional perspective.

8.4 A Closer Look at Microworlds as Embodiments of Mathematics

In examining the issues addressed up until this point, and in reviewing our quest for a "definition" of microworld, we may ask ourselves whether it is best to characterize a microworld in terms of a specific list of structural features, or to take a broader view which focuses on the experimentation-feedback cycle of learning typical of these environments. If we take the second approach, we may find included in the category of microworlds certain artifacts which are *not* based on programming languages or personal computers. For example, can we consider a calculator a "microworld"? An article by a mathematics teacher in a recent issue of the California Mathematics Council newsletter (Preibisius, 1993) describes a spontaneous activity within his classroom in which students investigated the factorial key (X!) on their calculators, after asking, "What's this button for?" Through their own series of experiments with the calculator, under minimal guidance from the teacher, these students were able to discover how this new mathematical entity, the factorial, functioned. In this case, the calculator was functioning not as it was designed, as a tool to automate well-understood mathematical operations, but instead as an exploratory environment for learning about new ones; that is, as a microworld. Depending on the level of experience and knowledge of the learner (as well as the sophistication of the calculator), a calculator can function as a medium for supporting exploratory learning of a range of mathematical entities and operations.

What about other structured domains? Can we consider an electronic keyboard, a piano or another musical instrument as a "microworld" for the structured environment of music? There seems to be a natural kind of exploration of music by young children who are presented with a musical instrument, long before the introduction of formal systems of musical notation or of the social meanings attached to the various genres of music (Noss, this volume, refers to a distinction between "inherent" and "delineated" musical meanings; children's experimentation-and-feedback cycles with musical instruments may perhaps be seen as examples of explorations of "inherent" musical meanings).

Very simple concrete materials can constitute environments for exploratory learning. Knots, for example, have been used as an arena for the exploration of topology, not only by adult mathematicians, but by 10- and 11-year old children. Strohecker describes their "double offering" as microworlds:

> Each knot is...its own universe, which invites contemplation of its topology both as it is being formed and as a completed object. Additionally, different knots are often quite similar, so that understanding something fundamental about one can lead to an understanding of another. In this sense, knots as a category of objects can be a microworld for learning about topology. (1991, p. 215)

There exists a range of concrete manipulable materials which could be viewed as microworlds, if we take the functional rather than the structural perspective. For example, Dienes blocks can be seen as "microworlds" for integers, or alternatively, for fractions, depending on the context in which they are introduced, on the initial assignment of meaning made by the teacher, on the problems and tasks carried out with them, in short, on the activity setting within which they are used.

The fact that the same artifact, whether a calculator or a set of Dienes blocks, can be utilized as a microworld for exploring more than one kind of mathematical entity, points us toward a deeper examination of notions of representation and meaning. We need to examine underlying assumptions about how computer environments and other symbol systems "represent" or "embody" ideas. In the remaining sections of this paper, we turn to issues of meaning, representation and "embodiment," using tools from philosophy, linguistics, sociology and psychology.

8.4.1 Microworlds as "Representations"

At this point, we will examine the general question of how microworlds can be said to represent mathematical, scientific or other ideas. To begin this examination, let us look again at various definitions and characterizations of microworld collected in this paper. For instance, there is the definition offered by the author, in which a microworld is described as "an environment, based in a computer or another medium, in which the central objects and relations of a domain are instantiated into a concrete or semiconcrete form" (Edwards, 1991, p. 123). Hoyles and Noss describe features of a computer medium which are "representations of mathematical structures and relationships which the learner is expected to abstract through interaction in the microworlds" and state further that "microworlds can be viewed as computational environments that 'embody' mathematical ideas" (1993). Similarly, Thompson talks of a graphic display and a set of operations constituting "a model of the concept or concepts being proposed to the student," and states that "in a very real sense, the microworld embodies the structure of the concept" (1987, p. 85). Dugdale speaks of the mathematics being "intrinsic" to a "working model" the students are given to explore. Clearly, it is common to talk about microworlds as "embodiments" of subdomains of mathematics or science. But exactly what is meant by these notions of "instantiation," "embodiment" or "modeling?" In what sense can we say that a mathematical concept or structure is "intrinsic" to a particular external representation? These questions take us into philosophical territory, but I think it is important in the current discussion to at least raise and address some of the issues relevant to a casting of microworlds as "representations."

8.4.2 Do Symbols or Microworlds "Carry" Meanings?

The general question we must address here concerns the notion of representation itself, and of the ways in which we think and talk about symbols and their meanings. In mathematics, notions about how symbols represent mathematical ideas, and about the relationship between the objects of mathematical thought and their expressions have been addressed by researchers and philosophers of mathematics alike (cf., Janvier, 1987; Kaput, 1987, 1991; Kitcher, 1984; Pimm, 1987; Skemp, 1987). Without tackling the question of the locus of reality of mathematics itself (cf., White 1956), we can at least acknowledge that external representations of mathematics are socially-constructed systems. In many cases, the constructions of these systems of representation has taken place over centuries, and some would state, as does Kaput (1987), that attention to representations is a central focus of mathematical activity, that "much actual research in mathematics is an attempt to extend... representations to new mathematical domains involving more esoteric objects" (p. 25). Although to the student being introduced to "school mathematics," it may seem that representational systems are fixed and unchanging (and indeed, as Kaput also points out, external representations which have become dissociated from their referents are a source of confusion to learners in mathematics), the creation of computer-based models for mathematics and science provide us with an opportunity to consider, design and evaluate new representations.

But we must ask ourselves what is it that we believe we are doing when we "embody" a subdomain of mathematics or science in a microworld? The language of "embodiment" makes it easy to believe that what we are doing is creating a representation which gives concrete and unambiguous form to the non-corporeal "stuff" of mathematics or science, through the creation of a "container" or carrier for ideas. It is common to speak of notations and representations as "carrying" meanings. For example, Kaput talks of "the differential ability of certain representations to *carry* certain aspects of the arithmetic of rational numbers better than other representations" (ibid., p. 19, emphasis added). Pimm states that "mathematical ideas are often *conveyed* using specialized, highly condensed symbol systems" and discusses these symbols as "efficient means of *storing* and *conveying* information" (1987; emphasis added). Skemp uses the following language to characterize symbols: "A symbol is a sound, or something visible, *mentally connected* to an idea" (1987, p. 47; emphasis added), and goes on to state, "This idea is the *meaning* of the symbol" (op. cit., emphasis in the original).

These familiar and natural ways to talk about symbols and meaning reveal an underlying metaphor[6] which also appears in certain ways of talking about microworlds as representations. This metaphor can be seen as a specialization or extension of a more general metaphor which structures our understanding of language itself. This system of implicit assumptions has been labeled the "conduit metaphor" for language and meaning (Lakoff and Johnson, 1980; Reddy, 1993). In the conduit metaphor, words or symbols (and in the current case, computer microworlds) are understood as containers for meanings. Lakoff and Johnson spell out the essential components of this metaphor:

> IDEAS (OR MEANINGS) ARE OBJECTS.
> LINGUISTIC EXPRESSIONS ARE CONTAINERS.
> COMMUNICATION IS SENDING.
>
> The speaker puts ideas (objects) into words (containers) and sends them (along a conduit) to a hearer who takes the idea/objects out of the word/containers. (1980, p. 10)

This metaphor helps us make sense of the language which is often used to talk about mathematical symbols and systems of representation: If mathematical ideas are objects, then the symbols and representations we encounter or create for them, whether on the computer or not, are like containers, which "carry" their meanings to the "hearer" (or "reader," or "viewer" of the computer screen). This is a very familiar and ubiquitous way to talk about mathematical symbols, but it has entailments that we may not wish to embrace. For instance, if linguistic or other representations are containers for meanings, then the metaphor implies that the meanings carried *by* the container have a fixed nature, regardless of context. Lakoff and Johnson point out this entailment, noting that "the MEANINGS ARE OBJECTS part of the metaphor...entails that meanings have an existence independent of people and contexts" (1980, p. 11.) This entailment runs counter to a sociological and philosophical perspective that holds that meanings are socially-constructed and maintained, even within domains such as mathematics and science (Berger and Luckman, 1966; Bloor, 1976). A further entailment or consequence of the conduit metaphor is that it implies that there may exist, somewhere, an ideal container, a representation that can perfectly link specific notations to their referents in mathematics and/or science. This implication is consistent with an objectivist approach to meaning, which holds that "there is an objectively correct way to associate symbols with things" (Santambrogio and Violi, 1988, p. 17). Designers of learning environments and cognitive scientists may believe that their objective is to find or build such ideal symbol systems.

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[6]"Metaphor" is used in a very specific sense here. Lakoff and Johnson have developed a theory, or a cognitive semantics, which holds that much of everyday language is metaphorical, in the sense that underlying conceptual mappings structure how we think about a wide range of concepts, and that these mappings are revealed in the language we use. For further details on the theory, see Lakoff, 1987, 1988, 1993; and Lakoff and Johnson, 1980.

I would like to warn against these assumptions, and the philosophical stance toward meaning which they reflect, when we consider microworlds as "representations" of mathematical ideas. If we decide that our task is to select or create a set of representations that we will be certain, *a priori*, will "carry" our intended meanings to the students, or that will "reify" an idealized mathematical domain, there is a risk that we will ignore the social aspects of meaning construction that can only be observed (or inferred) when a microworld is actually *used* by its target audience. It is common to talk of the "transparency" or "opacity" of representations; again, the "container" metaphor applies: The implication is that certain containers more easily allow the meanings contained therein to "show through." However, Meira, among others, has noted that whether a display is "transparent" (in the sense of its meaning being clear to the learner) depends to a great extent on the activities within which its use is situated. He states, "the transparency and efficiency of material displays in mathematics learning cannot be read from the objects themselves, but it is a constitutive aspect of the activity structures in which they are used" (1991, p. 22). When a designer constructs a computer microworld to represent some corner of mathematical or scientific knowledge, there is no guarantee that the user will see what the designer intended or what the designer sees—the "transparency" of the representation for the expert who creates the microworld is a function of his or her existing knowledge and participation in a community with shared understandings (Cobb, Yakel and Wood, 1992). The meaning of a computer microworld is not "carried" to the learner by the designers' representational choices, but is individually constructed and socially negotiated, in the course of solving problems and carrying out activities within a specific social context.[7]

The point of this discussion is simply, once again, to emphasize the importance of a process of design of representations and activities for mathematics teaching which is sensitive to and makes use of empirical information about their context of application (their "situatedness"). It is also to highlight a stance toward representation which holds that, "It is simply illusory that 'one can in language, or mental representations, or programs escape from the world of symbols to some formal but non-symbolic realm that confers significance,'" (Wilks, quoted in Santambrogio and Violi, 1988, p. 16). The meanings which we hope students will gain from working with computer environments for mathematics are, as are other meanings surrounding them, constructions of a particular community (Lave and Wenger, 1991; Resnick, 1991); they are not inherent in the computer environments themselves. It is our task as educators, I believe, to not only to design material artifacts which can support our students' further understandings in the domain, but also to investigate and act as resources for the engineering of effective social contexts for meaningful learning, in classrooms and other situations.

[7] Editor's note: See Hancock, this volume, for a somewhat complementary view of transparency as regards designed artifacts.

8.5 Conclusion

By way of conclusion, I would like to note one way in which the language used to talk about computer microworlds can be seen to be very apt and metaphorically powerful. We talk about microworlds as "embodiments" of mathematical or scientific ideas. It seems that computer environments, whether in educational, entertainment or work contexts, are increasingly being used as extensions of the capabilities of people, as evidenced by the following description of research in the area of molecular chemistry:

> At the University of California, San Francisco, Thomas Ferrin, Director of Computing, Molecular Graphics, recently commented, "If you have an idea you wish to pursue—for example, you may have a hunch the molecule will fit better if you rotate it and push it a little bit differently—you need instantaneous feedback. It's all part of the intellectual and creative process which occurs when you're investigating the structure and function of molecules." One of the key features that...workstations provide is the ability to very rapidly generate images of molecules that can be manipulated in real time, that is to say when the scientist moves the mouse, the molecule responds instantaneously. So the system forms, if you will, an extension of the scientist's hand and when he [sic] moves his hand he can see the molecule move. (Cruickshank, 1993, p. 14)

It is perhaps in this sense that we can speak of a microworld as "embodying" mathematics: not because of some reifying link between the representation and the mathematical entity, but because of the opportunity that such environments provide for learners to kinesthetically and intellectually interact with the designers' construction of a system of mathematical or scientific entities, as mediated through the symbol system of a computer program. The design of computer environments and other material displays, as well as the refinement of effective activity structures which can powerfully extend our students' understanding of mathematics and science represents a continuing challenge to educators and researchers.

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