The Construction of Meanings *In* and *For* a Stochastic Domain of Abstraction

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ABSTRACT

This study takes as its focus young children's intuitive knowledge of randomness. Previous work in this field has studied the misconceptions that people, especially adults, hold in making judgements of chance (see, for example, the work of Kahneman & Tversky and Konold).

In contrast, I study how primitive meanings for randomness form a basis for new meanings, a process which the misconceptions approach fails to illuminate. The guiding principle for this study is that the observation of students' evolving thought in a carefully designed computer-based domain will provide a better understanding of how the specific features of the domain shape and are shaped by activities within it.

There are, then, two deeply connected strands to this thesis: the study of children's evolving meanings for randomness as expressed *in* a computer-based microworld, and the articulation of design principles which encapsulate pedagogic meanings *for* that microworld. More specifically, the thesis aims to shed light upon the answers to four crucial questions:

Meanings for the domain

- What do formalisms of stochastic behaviour look like in a domain of abstraction?
- What structures in the domain for stochastic abstraction optimise the articulation of intuitions and the construction of new meanings?

Meanings in the domain

- What articulations of informal intuitions of stochastic behaviour do we observe?
- How do the structures of the domain support the forging of situated meanings?

The study uses an iterative design methodology, which cycles between the design of computer-based tools and the observation of children, between the ages of 9 and 11 years, as they use these tools. The thesis identifies initial meanings for the behaviour of various stochastic phenomena and traces how new pieces of knowledge, especially relating to long term random behaviour, emerge through the forging of connections between the internal and external resources.

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CHAPTER ONE

Perspectives on the Stochastic

1.1. OVERVIEW

This chapter begins with a personal account of how I became interested in the object of this study, and in so doing teases out some initial ideas for my investigation. It turns out that a look back at my personal involvement with stochastic ideas since childhood offers a first glimpse at the object and approach of the study. An analysis of the epistemology of the stochastic provides an entry point into a more detailed consideration of how the construction of meanings for the stochastic might evolve.

1.2. A PERSONAL ACCOUNT

This study brings together a number of themes which have occupied me throughout my life. This personal account will serve to explain my interest in the problem as well as outlining the nature of the problem. Towards this aim, I draw on memories .....

1.2.1. ..... as a Child

Around about the age of 11, I spent much of my time inventing games, which I would now call simulations. Like many of my male peers, I played a version of pencil cricket, in which the roll of the pencil determined runs scored and when the batsman was out. Even at that age, I was very dissatisfied with this crude game, as the frequencies of various outcomes did not reasonably represent the real world. There were, I can still recall, too many sixes and not enough catches. I spent many hours refining the game, using playing cards which offered more possibilities and therefore greater opportunities for sensitivity in the model, until there was a satisfying match between the simulation and real cricket. This match involved not just averages but also variation, as it was important to me even then that the unexpected should happen at a predictable long term rate.

As I look back at that time, I am struck first by the fact that I must have constructed, over a period of years, deeply connected stochastic meanings through building the games. I did not begin with these powerful ideas at the age of 11, or else I would have gone straight to the refined version of the game without all the iterations between. These ideas, which I have played with in other domains ever since, emerged through my game-construction activity. Secondly I am now aware that the level to which I took this activity was unusual, and that most of my peers were not engaged in the construction of new and better games but simply in playing the original crude version. So, as I look back at this period of my life, I wonder what it was that so fascinated me and how the activity of game construction might have been made more engaging for my peers.

1.2.2. ..... as a Teacher

Although as a school child, I had been offered virtually no education in probability, as an undergraduate, the experience was entirely formal, diametrically opposed to my own informal introduction to stochastic ideas. My entry to teaching coincided with the modern mathematics movement, which incorporated the teaching of probability to 11 year olds and above. The modern mathematics approach encouraged the use of experiments, such as rolling dice many times, in the hope that the children would somehow abstract the *patterns in randomness*. That this did not seem to happen except at a rudimentary level was frustrating, though in hindsight educational for me. This frustration could be expressed as a bewilderment as to what else I might offer these children, in order to optimise their opportunity to construct meaning for stochastic processes and events.

1.2.3. ..... as a Researcher

My research work has centred on the use of computers to support children's understanding of mathematics. In particular, I have spent a great deal of time working alongside children with Logo, and that has often resulted in the building of Logo microworlds. More recently I have become interested in the use of Boxer as a medium particularly tuned to the activity of microworld design. So one view of this research is that I am re-playing my childhood activity in an adult world. I am building an environment (rather than a simulation) in which (other) children can come across and use tools and structures, which are designed to offer them the opportunity to construct stochastic meanings. In so doing, I hope to gain some insights into how those tools and structures might support and change the learning process.

1.3. A PEDAGOGICAL ACCOUNT

The domain of probability points particularly sharply to a fundamental difficulty in mathematical pedagogy. On the one hand, it is deceptively close to everyday intuitions and experience, even language: chance encounters, random behaviours, likely occurrences. We could be forgiven for thinking that it is easy to build on these culturally embedded meanings, and that these would facilitate the transition to a mathematical way of thinking. Yet we know this is not the case; probability is a notoriously difficult topic, and it is often said that the only way for students to achieve satisfactory grades is to ignore altogether the relationship of probability to everyday notions of chance.

Perhaps more than in any branch of mathematics, the gap between potential everyday applicability and formal understanding is at its greatest in the domain of stochastics and probability. There are multiple opportunities to apply such knowledge as we go about our everyday lives; the playing of games with explicit random number generation (do I take the finesse of the queen of hearts?), in our sporting lives (should I adopt a long ball strategy?), and in our parenting (should I let my daughter walk to school?). Yet such opportunities do not seem to have led to widespread construction of meaning for stochastic concepts. On the contrary, the research reported in the next chapter suggests that adults' understanding of such ideas is often impoverished, even misconceived.

One basic explanation is that everyday experience fails to structure our intuitions to interpret those contexts as essentially stochastic because of a lack of validating feedback. For example, it has been suggested that the most fundamental of problem-solving strategies, that of trial and error, is simply not available in the case of probabilistic thinking. In the following quote, the authors refer to a choice between two wheels of fortune, where the chance of winning with wheel, X_1 clearly exceeds the chance of winning with X_2 (Borovnik & Peard, 1996):

> In fact, one can 'win' with a subsidiary strategy. The choice of the 'worse' wheel X_2 may lead to success; likewise the choice of the 'better' X_1, to failure. To convince someone who has chosen the 'wrong' wheel but has nevertheless won may not be such a simple matter. It can be counter-productive if the teacher tries to prompt students to rely on a trial-and-error approach, because this can lead to wild speculations which hinder the acquisition of adequate concepts.
>
> (p. 245)

In fact, mathematical discourse is simply *different* from everyday discourse, and the mathematical notion of probability is a scientific, rigorous concept in contrast to the fuzzy idea of chance which pervades everyday settings. But this simple statement masks the complexity of finding a pedagogical solution. In fact, we might argue that this complexity underpins a fundamental challenge of mathematical pedagogy: to construct situations which are rich in meanings for the learner, yet which point towards the specifically *mathematical* meanings which we would like them to acquire. As individuals make their way around their social and physical world, the intellectual tools at their disposal for mathematisation and abstraction are fairly impoverished. There is no need for them: everyday, pragmatic activity is adequately served by the fuzzy linguistic tools and artefacts that have emerged in the culture over millennia. Thinking mathematically demands more: it presupposes that one has a more or less rich pool of intellectual tools at one's disposal: algebraic notation, symbolism, and so on. These are precisely the intellectual tools one has at hand if one is a mathematician, and precisely those which one lacks if one has yet to be inducted into mathematical discourse.

This puts us in a kind of pedagogical loop: We would like people to gain access and power over these tools so that they can make mathematical abstractions. But, in order to make mathematical abstractions, it seems that they need access to precisely these tools.

1.4. AN EPISTEMOLOGICAL ACCOUNT

At the heart of the knowledge domain of stochastics lies a collection of linked ideas, which are spectacularly difficult to pin down; consider the late development of probability as a formally defined piece of mathematics[1] and the continuing controversies about their exact meanings (Hacking, 1975)

> ...we may readily confirm the fact that for all our advances in mathematical technology, a good many aspects of that dual concept of probability *(frequentist and subjective)* have been there from the beginning. The theories of today seem to compete in a space of possible theories that can be discerned even in the earliest years of our concept.
>
> (p. 16)

Each of these ideas in the continuing emergence of probability seems to declare its own epistemology. One strand, the *frequentist* view, regards probability as a limit of the proportion of outcomes to the number of trials as the latter tends towards infinity. Another strand sees probability as an attribute of the event, so that a coin will tend towards a 50% rate of landing on heads because it is simply in its nature to do so. A third school, the subjectivists, regard probability as a measure of belief, so that they might just accept an evens bet that the next toss of a coin would land on heads (suggesting that *their* probability for an outcome of a head is one half) but they would probably accept with enthusiasm an evens bet that they would cross the road safely (indicating a probability of crossing the road safely of above one half). Bayesians, on the other hand, regard probability as a rational measure of belief based on one's knowledge of the *a priori* probabilities and the conditions of the experiment.

These strands in the development of the concept of probability can lead to different perspectives on some central ideas in this domain.

1.4.1. Randomness and the Law of Large Numbers

Randomness as a concept is difficult to pin down. There are at least four perspectives on randomness.

Informal views of randomness

Informal everyday views of randomness seem to incorporate intuitions of arbitrariness, unpredictability, even fairness. These ideas seem to reflect how randomness behaves in the short term. In contrast, informal views of long term behaviour are often expressed irrationally, or so it seems to the statistician. We hear people speak of the "Law of Averages", which seems to suggest that somehow randomness redresses imbalances in outcomes in the longer term. In this view, a soccer team on a losing streak is likely to win today because of the *Law of Averages*. After a series of poor hands whilst playing Bridge, this law would suggest that I might now expect a splendid hand to make up for all the weak hands that have gone before. Such ideas seem to pervade our culture.

Formal view of randomness

In contrast, there is a formal mathematical view of randomness based around the notion of a random variable (Kolmogorov, 1950), as a function $x(\xi)$ which maps set E (of events) into the set R^1 of all real numbers. In the definition below, the system of sets $\mathfrak{I}$ is essentially the set of subsets of the events in E.

> A real single-valued function $x(\xi)$, defined on the basic set E, is called a *random variable* if for each choice of a real number a the set $\{x<a\}$ of all ξ for which the inequality $x<a$ holds true, belongs to the system of sets $\mathfrak{I}$.
>
> (p. 22)

It has been suggested to me that this definition contains an error, though the quotation is correct. For my purposes, the obscurity of this formal definition to most readers, whether correct or not, only serves to emphasise the stark contrast between this view and other views of randomness.

The formal view of long term behaviour of a random variable is summed up in the fundamentally important Law of Large Numbers (Kolmogorov, 1950):

> The random variables S_n of the sequence
>
> $$S_1,\ S_2,\ \ldots,\ S_n,\ \ldots$$
>
> are *strongly stable* if there exists a sequence of numbers
>
> $$d_1,\ d_2,\ \ldots,\ d_n,\ \ldots$$
>
> such that the random variables
>
> $$S_n - d_n$$
>
> almost certainly tend to zero as $n \to +\infty$.
>
> (p. 66)

Let us interpret this definition in the straight forward case of throwing a dice. Successive throws of a dice can be seen as a sequence of random variables, each representing one throw of the dice. Consider the random variable, X_i ,which takes the value 1 when the throw is a 6 and 0 otherwise. The proportion of sixes after n throws, S_n, can be computed as $\sum_{i=1}^{n} \frac{X_i}{n}$. Ongoing values of this proportion for successive throws make up the sequence, $S_1, S_2, \ldots, S_n, \ldots$ in the above definition. Now let $d_1, d_2, \ldots, d_n, \ldots$ be the sequence $\frac{1}{6}, \frac{1}{6} \ldots \frac{1}{6}$. The Law of Large Numbers implies that the difference between the two sequences, S_n and d_n , almost certainly vanishes as the number of throws, *n,* becomes large.

Expressed loosely this law proposes that statistics of larger samples (e.g. means) more closely approximate the corresponding parameters of populations and are thus less variable than those of smaller samples.

Pseudo random numbers

Another perspective on randomness has evolved as a result of the need to incorporate random variation into computer-based models. Random numbers as generated by a computer are constructed according to a complex, highly variable, sequence. The sequence is seeded by attributes of the computer at the moment of generation. If the function that generates the sequence and the seed are known then the pseudo-random number is entirely predictable. A formal view of randomness would therefore not regard these numbers as random at all. However, an informal view may well do so since in practice the function and the seed are not likely to be known and so the pseudo-random numbers possess the sorts of properties regarded intuitively as random.

Developments in artificial intelligence have concerned a notion referred to as *complexity.* The complexity of a pattern is a measure of how much memory is needed to encapsulate that pattern. A random pattern in the formal sense could be said to be infinitely complex. Pseudo-random numbers are not infinitely complex.

A modelling view of randomness

A modelling view of randomness would also find the notion of pseudo-random number as unproblematic. In this perspective, randomness is a model which an individual can use to describe a real situation. Thus, the outcome of tossing a coin might be conveniently modelled in stochastic terms since it is unlikely that the next outcome is predictable and yet in the long term one might expect roughly half of the outcomes to be heads. The model will have limitations; for example, the person tossing the coin may intentionally bias the toss perhaps by ensuring the coin only flips once.

Just like the generation of pseudo-random numbers on a computer, the model fits the situation well provided little is known about the experiment. In contrast, the coin tossing could be seen in terms of a deterministic model in the sense that if we knew enough about the various parameters that determine the outcome, we could predict the result.

1.4.2. Stochastic or Deterministic

The modelling perspective on randomness has introduced an important issue, whether events are conceived as stochastic or deterministic. Shall we regard the outcome of the toss of a coin (head or tail) as intrinsically different from, say, the outcome of crossing the road (with injury or with safety)? There is a duality in our current conception of probability which has been present throughout its historical development. On the one hand probability is used as a limit of the proportion of successes to the number of trials, whereas on the other it is used in a subjective manner, as a measure of belief. A strict frequentist view would seek to explain events such as crossing the road in frequentist terms even though such an event would be difficult to replicate in practice. This pragmatic obstacle may lead to a tendency to separate coin tossing from road crossing so that coin tossing is seen as essentially stochastic whereas road crossing is seen as deterministic (causal factors might be the choice of crossing point or the speed of oncoming traffic). As we have seen, we should leave room for a deterministic view of the coin in which we can conceive, even though we can not isolate and identify, all the innumerable factors which lead to the coin's outcome. Equally, a stochastic version of crossing the road, in which one envisages a certain chance of successfully crossing the road, may require us to entertain a more subjective perspective on probability. In a modelling perspective on randomness, the *or* in the title, 'stochastic or deterministic', can be constructed as an inclusive 'or'.

This view of randomness has also been put forward by Biehler (1994):

> A major point is that the ontological debate of whether something 'is' deterministic or not may not be useful, rather, a situation can be described with deterministic and with probabilistic models and one has to decide what will be more adequate for a certain purpose.
>
> (p. 4)

1.4.3. Distribution

At the centre of any epistemological analysis of the stochastic must be the notion that such events may be unpredictable in their outcome in the short term but that the relative frequencies of the various outcomes in the possibility space tend towards some limiting values as the number of trials tends towards infinity. These limiting proportions or probabilities constitute the distribution. The notion that there is such regularity amid all the apparent unpredictability is fundamentally important.

In particular, the uniform distribution, in the discrete case, represents equally likely outcomes. This distribution reflects one view of *fairness*. We will see that fairness and randomness are often intertwined at an intuitive level, so it is worth stressing here that formal, pseudo and modelling views of randomness all recognise distributions other than uniformity. The separation of fairness and randomness is an important development (see the *equiprobability bias* in the next chapter).

1.5. RESEARCH THEMES

This study brings together the themes presented in this chapter. My interest in designing simulation games as a child, and in constructing microworlds for the learner as a researcher, finds expression through the building of an environment in which children can themselves articulate their views of randomness. My experience as a teacher suggests that a formal approach to randomness is unlikely to connect with children's everyday informal intuitions. Previous research will be presented in the next chapter which supports this view.

The study will examine the conjecture that young children can discover that there exist new perspectives on randomness through interaction with computer-based tools. The general approach will therefore be one in which randomness is a model for the behaviour of phenomena, a perspective which offers new insights to be gained, especially with respect to long term behaviour. Throughout this study, there will be reference to the behaviour of stochastic phenomena, by which I mean to suggest that the behaviour of such phenomena is at least potentially amenable to a stochastic interpretation. It is *not* my intention to suggest that the phenomena are in themselves stochastic.

Whilst pursuing the broad theme of studying children's intuitions of the stochastic, I aim to understand how such intuitions are shaped by the tools offered. It is my intention that this aspect of the study will develop some rich insights into the principal characteristics of such tools.

1.6. STRUCTURE OF THESIS

This chapter then has offered a first glimpse at the issues at the heart of this study. Chapter Two will examine in some detail previous research in this field. Contrasting approaches to the study of people's understanding of chance will be reviewed and some important gaps in the body of knowledge will be identified.

A number of specific aims for this study will emerge out of this survey and direct the path of the remainder of the thesis. These aims, including a sketch of the approach to their study, will be discussed in Chapter Three. The method that will be used in the research will be detailed in Chapter Four, and the methodological issues raised will be discussed.

Chapter Five begins the reporting on the data by focusing on the early phases of the research. This chapter will trace the interweaving of the two main themes, the design of the computer-based tools and the children's articulation of meanings for the stochastic as they use those tools.

The final phase of the research will be presented in Chapters Six through to Nine. The first of these chapters will set out the final design of the software which emerged from the earlier phases and discuss the main features of the tools. Chapters Seven to Nine will draw on a number of case studies in which children used these tools to discuss their construction of meanings for the stochastic. Chapter Ten will summarise the findings in the previous four chapters and discuss the wider implications of this research in terms of the development of theory.

NOTES

1 According to Hacking (1975), the first limit theorem was set out in Ars Conjectandi by Bernoulli in 1713. Essentially this theorem proves the weak law of large numbers : the probability of an n-fold sequence in which $|p - s_n| < \varepsilon$ increases to 1 as n grows without bound.

CHAPTER TWO

A Review of the Literature

2.1. OVERVIEW

The survey of literature is in three parts. Part One begins by considering previous work on understanding of probability and chance. This work is shown to have several shortcomings as far as the study of the construction of meanings for stochastic phenomena[2] is concerned. In an effort to locate stochastic meaning, Part Two continues by looking at the wider research on intuitions and conceptual change, including the specific role of external social and environmental factors. By considering an alternative view of abstraction consistent with the emerging view of conceptual change, a definition of a domain for stochastic abstraction is proposed. In Part Three, abstraction is interpreted in the case where abstracting takes place within a computer-based setting, and the contribution of technology towards the construction of meanings is considered.

PART ONE

2.2. OPERATIONAL THINKING AND PROBABILITY

Piaget and Inhelder proposed in their seminal work (Piaget & Inhelder, 1951):

> As soon as chance is discovered as indeterminate relationships not composable by operative methods and, contrary to all the operations, irreversible, the mind seeks to assimilate this unexpected obstacle encountered on the road to developing deduction. There is simply one manner of understanding and explaining this: Group the relations at play according to the model of the operative systems. If chance for a moment makes reason useless, sooner or later reason reacts by interpreting chance, and the only way of doing this is to treat it as if it were, in part at least, composable and reversible, that is, as if one could try to determine it in spite of everything. It is from this need that probabilistic composition is born..."
>
> (p. 230)

In Piaget and Inhelder's vision, the construction of probabilistic knowledge is contingent upon the construction of both combinatorial and proportional mental systems. A critical step in this process is the construction of a notion of mixture. When the child can conceive of the various ways in which objects can be combined, he or she is able to recognise how mixtures are combined and how their mixing might be reversed. When the child is confronted with a random mixture, Piaget and Inhelder argue that an explanation will be constructed in terms of many such combinations, but at some point the child will recognise that the mixture is in practice irreversible. We might imagine a child watching the numbered balls inside a lottery machine colliding over and over again until the machine somehow chooses the winning balls. Despite repeated efforts to explain why those particular balls were chosen, such explanations are doomed to failure. And in the end, so the argument goes, the notion of multiply determined causes and effects on individual balls is replaced by one of irreversibility: ultimately, the child comes to realise that the actions which conspired to determine which balls were chosen cannot be identified.

For Piaget and Inhelder, the child in such a situation is in a position of conflict, since all operations previously constructed have been composable and reversible. Yet here is a situation which is apparently neither composable nor reversible. At this stage, therefore, the child may recognise random mixtures as those whose structure cannot be explained by operational understanding, a state of affairs which is not resolved until formal thought allows the concept of proportionality. Then, and only then, the child is able to construct a formal system of probability and the conflict is resolved. The key breakthrough is that the construction of probability enables the child to treat the random as if it were operational.

This formulation neatly highlights a central issue: the child first recognises that a phenomenon is stochastic *before* he or she is able to operationalise it. But the Piagetian framework does not explain how notions of irreversibility manifest themselves. How is it that children comes to interpret stochastic phenomena at a time when determinism is their main, if not their only, way of looking at the world? Of course a strict Piagetian answer would be "through accommodation". But this only names the process, it hardly explains it. If accommodation happens, how does it happen? How does the child resolve the paradox and how can intuitions based on deterministic thinking serve as the basis for making sense of stochastic processes which seem, by their very essence, to demand a rejection of the deterministic as a precondition for their understanding? And crucially, is the accommodation process essentially developmental, independent of the tools and resources available for the expression of the phenomena the child is trying to describe?

Piaget's approach is one of genetic epistemology, which takes a developmental view of knowledge and examines the relationship between the macro-history of cultures and species and individual development. In his view, the construction of operations is a universal phenomenon and so his study of the origin of chance, as part of this monolithic theory development, must be explained in terms of operations. One does not have to refute Piaget's stance to suggest that other approaches may take different slices through this complex field to gain a fresh picture of how stochastic knowledge develops.

Whereas Piaget integrated his observations of the origins of the idea of chance into a general theoretical framework, a huge and very influential body of literature developed during the 1970's and 1980's, examining the heuristics that people (mostly adults) use to make judgements of chance. The approach was essentially atheoretical and presents a very different picture from that of Piaget.

2.3. HEURISTICS IN THE JUDGEMENT OF CHANCE

One of the outstanding contributions to the study of how people make judgements of chance has been that of Daniel Kahneman and Amos Tversky. They have identified a number of heuristics; I will describe two, *representativeness* and *availability*, in detail to give a clear picture of their approach. (An overview of this work can be found in: Kahneman & Tversky, 1982; Tversky & Kahneman, 1974)

2.3.1. The Representative Heuristic

When using the representativeness heuristic, people predict the outcome which appears to be the most representative of the evidence (Kahneman & Tversky, 1973; Tversky & Kahneman, 1983). In many situations, such a heuristic leads to valid conclusions, experiences which confirm the heuristic as an appropriate strategy. However, Kahneman and Tversky showed that, at least under certain experimental conditions, people's use of the representativeness heuristic can lead to bias.

For example, they report on how this heuristic can lead to situations in which the conjunction of two events is seen as more likely than either of the constituent events. They gave a group of undergraduates this thumbnail sketch (Tversky & Kahneman, 1983):

> Bill is 34 years old. He is intelligent, but unimaginative, compulsive, and generally lifeless. In school, he was strong in mathematics but weak in social studies and humanities.
>
> (p. 297)

The subjects were then asked to rank eight statements according to the extent to which they resemble the typical member of that class and secondly according to the probability that the statement was true. The subjects placed conjunctions such as "Bill is an accountant who plays jazz for a hobby" as both more representative and more probable than the constituent events, "Bill is an accountant" and "Bill plays jazz for a hobby".

Whilst it is acceptable that the conjunction should be regarded as of greater descriptive value and so more similar to the thumbnail outline of Bill, it is surely unacceptable that a conjunction of two events can be regarded as more likely than either of the events which make up that conjunction, in contradiction to one of the basic laws of probability, and indeed common-sense.

Kahneman and Tversky have identified a number of other variations in the way that the representativeness heuristic can lead to bias. I describe those cases which seem to have more relevance to this study:

Insensitivity to prior probability of outcomes

The tendency to look for similarities between outcome and evidence can be so strong that subjects often ignore base-rate frequencies or prior distributions in making their judgement. For example, Kahneman and Tversky gave the following question to a group of subjects (Kahneman & Tversky, 1973):

> A panel of psychologists have interviewed and administered personality tests to 30 engineers and 70 lawyers, all successful in their respective fields. On the basis of this information, thumbnail descriptions of the 30 engineers and 70 lawyers have been written. You will find on your forms five descriptions, chosen at random from the 100 available descriptions, For each description, please indicate your probability that the person described is an engineer, on a scale from 0 to 100.
>
> The same task has been performed by a panel of experts, who were highly accurate in assigning probabilities to the various descriptions. You will be paid a bonus to the extent that your estimates come close to those of the expert panel.
>
> (p. 241)

A similar question was given to a second group except that this time the number of engineers in the question was 70 and the number of lawyers was only 30. In fact, the prior probabilities were largely ignored by most subjects, so that similar conclusions were drawn by both groups. Kahneman and Tversky give this and similar data as evidence that the subjects are using a representative heuristic in making their judgements and this largely dominates any sensitivity to prior probabilities (except when no description is given in which case the subjects make correct use of the prior probabilities).

Insensitivity to sample size

When people are required to make judgements about the probability of an outcome involving a sample, they will tend to refer to the population as a whole and ignore the sample size. For example, the following question was given to a group of undergraduates (Tversky & Kahneman, 1974)

> A certain town is served by two hospitals. In the larger hospital about 45 babies are born each day, and in the smaller hospital about 15 babies are born each day. As you know, about 50 percent of all babies are boys. However, the exact percentage varies from day to day. Sometimes it may be higher than 50 percent, sometimes lower.
>
> For a period of 1 year, each hospital recorded the number of days on which more than 60% of babies born were boys. Which hospital do you think recorded more such days?
>
> (p. 1125)

Most subjects judged the probability of obtaining more than 60% boys to be the same in the two hospitals. Kahneman and Tversky regard this as evidence of the representativeness heuristic; the events are equally likely since the data given is equally representative of the general population.

Local representativeness

People believe that a sequence of events generated stochastically will represent the essential characteristics of that process, even when the sequence is quite short. Kahneman and Tversky refer to this phenomenon as *local representativeness.* So, in a sequence of coin tosses, the outcome HTHTTH is regarded as more likely than HHHTTT and also HHHHTH since the former better represents (locally) the two outcomes H (head) and T (tail). Similarly, they explain the well-known gambler's fallacy as an example of local representativeness. Thus, when faced on the roulette wheel with a long sequence of *red* outcomes, most people predict that a *black* is somehow overdue and is therefore more likely to occur on the next spin of the wheel.

Falk has suggested that the gambler's fallacy is the consequence in difficulties of encoding rather than the use of a representativeness heuristic (Falk & Konold, 1997):

> The judgement of randomness results from the repeated failure of attempts to find regularities (confirm hypotheses; detect encoding shortcuts).
>
> (p. 316)

She borrows from a definition of randomness used in artificial intelligence where the degree of randomness is measured by how many bits are needed to encapsulate the pattern (Falk & Konold, 1994). According to this analysis, random sequences are difficult to encode because they resist encapsulation. Though this work, and her earlier work from which it has grown (Falk, 1981), is interesting, I do question whether approaches which present to subjects static lists of digits lose some validity by not allowing those subjects to draw on possible intuitions of randomness as a dynamic process, in which conjectured patterns might be tested for sustainability by the generation of further digits.

There are in fact everyday settings which seem to support a representativeness type of thinking. Consider this setting for the use of local representativeness. If one walks into a party early, and finds six women there, it is a good bet that the seventh will be a man. Experience leads us to believe that parties often have approximately the same number of men and women. So everyday reasoning leads to the (mostly correct) conclusion for parties, but one which is false in general: and false in the sense of roulette. Everyday settings are generally unhelpful in supporting recognition of the limitations of the representativeness criterion.

Worse still, everyday life can be downright misleading: playing roulette, it doesn't actually matter if one makes the false assumption: in reality I am as likely to win if I put my money on black as red — there is no penalty for my "wrong" action. Everything conspires to make me believe that which isn't mathematically true.

2.3.2. The Availability Heuristic

When using the availability heuristic, people predict the outcome which is the most easily brought to mind (Tversky & Kahneman, 1973; Tversky & Kahneman, 1983). They judge the likelihood of an event by how many cases they can recall. Although in many situations, this strategy may lead to valid conclusions, Kahneman and Tversky (1973) showed that, like the representativeness heuristic, the availability heuristic can also lead to bias. I describe those cases of bias which seem more relevant to this study:

Biases due to the effectiveness of a search set

Tversky and Kahneman (1983) described how students from the University of British Columbia found it easier to list words ending in _ _ _ _ i n g than _ _ _ _ _ n _. Similarly, predictions of how many such words would exist in four pages of a novel favoured words ending in *ing*. As with representativeness then, the availability heuristic can lead to incorrect evaluation of conjunctions. The relevance here to the judgement of chance is the claim that one strategy for making such judgements is to search for occurrences of the event. If the method of searching is fallible (as indicated above) then the judgement will contain bias.

Biases due to retrievability of instances

A class whose instances are easily retrieved will appear more numerous than one which is not. Kahneman and Tversky (1974) describe one experiment which serves to illustrate this case quite clearly. Subjects heard a list of well-known people. In some cases, the list contained relatively more famous men whereas in other cases the women were more famous. Subjects judged that the list contained more men or women according to whether the men or women were better known rather than their actual frequency.

Biases of imaginability

The imaginability bias due to the availability heuristic occurs when the class to be judged does not already reside in memory but has to be generated according to a rule. However, the availability of the class will be biased by how easily it is constructed. Consider the following exercise (Tversky & Kahneman, 1974):

> Consider a group of 10 people who form sub-committees of k members, $2 \leq k \leq 8$. How many different committees of size k members can be formed?
>
> (p. 1127)

It is easier to imagine small committees than large ones so they appear more numerous even though clearly the number of committees of size k equals the number of size 10-k.

Illusory correlation

The judgement of how frequently two events coincide can be based on the strength of the associative bond between them. However, this bond may in fact be an illusory correlation, perhaps established in folk lore. Naive judges were presented with data about several hypothetical mental patients. The data consisted of a clinical diagnosis and a sketch of a person made by the patient. The judges were asked to estimate how often a particular diagnosis, such as paranoia, had been accompanied by a physical trait, such as peculiar eyes. The co-existence of such factors was far higher in their judgements than was actually the case in the data. According to Tversky and Kahneman, the subjects had drawn on associations in their mind planted by clinical lore, making them *over-available*.

This type of thinking serves us well in many everyday settings. I decide to go shopping now rather than tomorrow because recent memories lead me to a prediction that the supermarket will be quite empty at this time of day. If the prediction turns out to be wrong and the shopping experience is particularly awful, I may give this event undue weighting when it comes to making similar decisions in the future.

2.3.3. The Equiprobability Bias

Lecoutre found a tendency for people to assume that the different outcomes occupying the possibility space were equally likely, even when the individuals involved were grounded in probability theory (Lecoutre, 1992). Hence there would be a tendency to assume that, on shaking two dice, a total of 9 was just as likely as a total of 7, each total being *just a matter of chance*. Apparently such notions of equiprobability are resistant to modification.

When an 'equivalent' test item, in which the chance element was removed leaving however the same combinatoric calculations, was given to Lecoutre's subjects, the equiprobability bias significantly diminished. The equiprobability bias therefore seemed to be closely linked to their understanding of chance.

2.3.4. The Outcome Approach

Cliff Konold has found a tendency for people to interpret situations as non-stochastic when others, 'experts', might regard those same situations as chance-dependent. He has called this way of thinking, the "outcome approach" (Konold, 1989).When inclined to the outcome approach, people make predictions which are often based on causal factors and they tend to assign numbers as "probabilities" on the basis of the strength of the perceived causal relationship[3]. More often, the probabilities are used as modifiers, with 50% meaning that no sensible prediction could be made. In common with bias due to the representativeness heuristic, the outcome approach disregards frequency information.

When using the outcome approach, people tend to treat the outcomes as either happening or not happening; they do not see the situation as demanding a probabilistic approach at all. Uncertainty is dealt with by predicting the outcome of the next trial and then evaluating the prediction as being correct or incorrect after a single trial. Consider this example. I wish to cross the road, which, after some rudimentary safety precautions, I do. Having just crossed the road, my attention is entirely focused on the fact that I crossed safely. I make no attempt to evaluate my road-crossing strategies. Perhaps I crossed from behind a car ignoring a nearby safe-crossing point. The tendency then is not to analyse the situation in terms of what might have happened with associated probabilities, but to focus on what in fact did happen. The outcome approach reflects a lack of awareness that there exists a stochastic dimension which could be considered.

I like the following example which will be appreciated by Bridge players the world over (though not if they themselves incline towards the outcome approach). I remember reading a humorous comment about slam bidding when, many years ago, I was trying to learn a new bidding system (Garozzo & Yallouze, 1969).

> One of the first essentials is to know when a slam is 'good'. For Mrs. Guggenheim any slam she makes is a good one, and if it involves a couple of finesses, so much the better.
>
> (p. 139)

The fictional Mrs Guggenheim would benefit by a more refined assessment of her slam bidding in which she considers, given the same state of knowledge about the bidding, how often the deal of the cards would lead to a successful slam. Such an approach would demand a stochastic perspective, in which a good bid of a slam might well lead to an unsuccessful contract, and a poor bid could in fact be successful. Mrs Guggenheim was clearly predisposed to the outcome approach.

The outcome approach is NOT seen as a belief system in that individuals will vary between this and other approaches but, according to Konold, some individuals do seem to be more likely to adopt this approach.

In the face of the above heuristics and predispositions, which betray the naiveté of stochastic intuitions even (especially?) amongst adults, teachers and students often engage in an avoidance strategy. Advice to students often goes something like this: "Ignore reality. Probability is just counter-intuitive. Always work with the definitions and standard methods.... If a trial may result in any one of n exhaustive, mutually exclusive and equally likely cases, and m of these are favourable to an event A, then the probability that A will happen as the result of the trial is measured by the quotient $\frac{m}{n}$." We erect a coherent symbolic edifice which a few will understand and most will not.

Given these likely inferences from the heuristic research, we need to consider carefully just how valid they are.

2.3.5. Criticisms of the Heuristics Research

I have given a detailed account of the research into the heuristics that we use in making judgements of chance. One reason for giving this work such prominence is that it has been very influential and makes up a substantial part of the literature in this field.

What are the implications of the heuristics research?

Two inferences could be drawn from the heuristics research, each reflecting a different interpretation of the nature of those heuristics. Either, people make these errors in making judgements of chance because:

(i) minds are somehow hard-wired to obey the intuitions that give rise, for example, to representative thinking; or

(ii) intuitions are insufficiently developed to deal with the tasks in an any more sophisticated way than that observed.

The first of these two inferences is the stronger, but either view has important implications for the teaching and learning of probability. The strong inference would suggest that we will only succeed by teaching probability as a formal subject, disconnected from any everyday intuitions. The weaker inference leaves open the question as to how we might set up more effective learning environments. The remainder of this chapter will examine and refute the first inference and begin to shed some light upon the second. I will begin by offering some specific criticisms of the heuristics research, each formulated as a question which the research fails to address.

What is the theoretical framework?

Kahneman and Tversky admit that their work is essentially atheoretical or, to be more precise, they believe that the systematic listing of such heuristics itself constitutes a theory (Kahneman, 1991).

> I take the distinctive feature of theory to be a commitment to completeness (within reason) and a consequent commitment to critical testing, in a specified *domain of refutation*, which is often quite narrow.
>
> (p. 143)

Tversky and Kahneman claim that they have several reasons for the focus on systematic errors (Kahneman, Slovic, & Tversky, 1982). They claim that their approach:

- exposes our intellectual limitations and suggests ways of improving the quality of our thinking;
- errors and biases reveal the psychological processes that govern judgement

and inference;

- mistakes and fallacies help the mapping of human intuitions indicating which principles are non-intuitive or counter-intuitive.

The first reason seems to be consistent with the weak inference and we might conclude that there is reason to believe that better learning environments could reduce the biasing effects of heuristics. The third reason presents a very static view of intuitions consistent apparently with the strong inference. I would suggest that it is the lack of a theoretical framework in the heuristics approach to research which leads to this confusion. If their aim is to reveal the psychological processes that govern judgement then a theoretical framework of change in intuitions would surely demand a different methodology from that employed. The imperative must surely be to look closely at the process rather than taking a snapshot of students' intuitions as seems to be the case in questionnaire/ interview methodology.

It remains an open question as to whether we should accept their data in a spirit which suggests that such heuristics are inevitable, or whether different intuitions might be constructed under different pedagogical circumstances. Indeed, further questions are raised by the heuristics research without answers to which the research lacks the explanatory value to which this study aspires.

Does the setting influence the shaping of intuitions?

The data for these studies was generally captured through questionnaire or interview in situations where the purpose for using a stochastic model is not always evident. Let me remind the reader of the task set by Kahneman and Tversky and discussed in section 2.3.1. in which students were asked to assess whether Bill was likely to be a jazz-playing accountant.

Students judged that it was more probable that Bill was an accountant who played jazz than that he was an accountant. Perhaps the students were using a stochastic model incorrectly, and the representativeness heuristic may be an appropriate way of labelling the students' strategies. On the other hand, perhaps the students did not see this as a task to which such a model should be applied. Perhaps some of the students recognised this as the sort of setting in which you meet someone for the first time and you wonder about their occupation. What matters in this context is not the logical distinction between an accountant and an accountant who plays jazz but the richness of the description.

The lack of a theoretical model means that we do not know whether we should expect the setting to be important or not. The methodological approach tries to eliminate the influence of setting by removing it from the task. Perhaps though setting is a very influential factor when making judgements of chance, in which case the heuristics research loses much of its validity if we try to apply it to everyday contexts. If our ultimate aim is to apply research findings to teaching and learning contexts[4], then it makes no sense to remove setting from the experiment. Indeed, if it were true that we could infer from the heuristics research that setting was not influential (and I do not see how we can possibly make this inference when setting was intentionally ignored) then we are in a position of no hope from a pedagogical perspective.

I will consider later and in more detail the importance of the setting on conceptual change.

When are we NOT fallible?

One of the weaknesses of the heuristic research is that it emphasises fallibility. We are left with an impression that human behaviour is essentially irrational. Lopes argues that, during the 1970's, the research of Kahneman and Tversky shifted the popular view from one of people as effective decision makers to one of ineffectiveness (Lopes, 1991). Lopes criticises the approach not because of the conclusion that people use heuristics to make judgements of chance but because of the implication that judgements are generally poor (though some have argued that the representativeness heuristic is not irrational at all (Cohen, 1979)). Lopes argues that the emphasis changes from one of identifying heuristics to the bias in those heuristics and therefore presents an unfair image of how often such bias occurs. According to Lopes, the language becomes evaluative rather than neutral.

The entire focus of the heuristics approach is on errors that people make when exercising judgement. It offers very little assistance as far as pedagogy is concerned. The emphasis on errors and irrationality makes more pressing the need to identify and examine more closely situations in which people do in fact seem able to use statistical approaches. This was the agenda for a series of studies which suggests three factors which, when present, seem to improve the likelihood that rational statistical reasoning will be used (Nisbett, Krantz, Jepson, & Kunda, 1983).

(i) Clarity of Sample Space and the Sampling Process

Randomising devices are designed so that the sample space for a single trial is obvious and so that repeatability of trials is salient. In the social domain this is not always the case; more often than not it is difficult to imagine any aspect of the problem as being repeatable. As a result, in the social domain, Nisbett et al claim that people tend to rely on the representativeness heuristic but in situations where the random generator is clear and defines a well-understood sample space, people are likely to use statistical reasoning.

(ii) Recognition of the operation of chance factors

Whilst it is relatively easy to recognise the random nature of simple devices like dice and playing cards, the random nature of other events can at times be obscure. However, even some social events are amenable to such an interpretation. For example, experience of a particular sport may allow one to recognise the fluctuations within play and in the results of such games. According to Nisbett et al, the random nature of social events may not be as explicit as that of randomising devices but they can nevertheless be discriminated.

(iii) Cultural Prescriptions

Nisbett et al argue that modern European young children are able to reason statistically in a way that mediaeval children probably could not because of the trickle down effect of cultural knowledge. Today there is an increasing tendency to talk statistically in social domains, especially sport, and this changing culture may have positive effects on children's ability to apply statistical heuristics in a wider range of situations in the future.

A major implication of these studies is that there should be a substantial domain specificity of statistical reasoning. Its use should be rare where: (a) it is hard to discern the sample space and process, b) the role of chance is unclear, and c) there is no cultural prescription for statistical reasoning.

Jacobs and Potenza also investigated the use of heuristics in different types of settings (Jacobs & Potenza, 1991). They found a clear distinction between social and object judgements, with a tendency not to refer to frequencies in the former situations . They also looked at the effect of introducing biasing information into the problem.

If we accept Nisbett's interpretation of their data, we are forced to conclude that the nature of the environment in which the child experiences stochastic events and processes may have a substantial positive effect on the use of statistical intuitions. By helping children to be more aware of the underlying random effects, and by building such experiences into the lives of young children, there is reason to believe that children may begin to use and construct statistical intuitions.

2.3.6. Criticisms of the Misconceptions Approach

Piaget had made it abundantly clear that children were not simply less informed versions of adults but thought in qualitatively different ways. The 1960's and 1970's were marked by a huge research effort in mathematics education to locate what became known as 'misconceptions', examples of misconceived or simply wrong knowledge or understanding articulated by children. These studies, (reviewed in Confrey, 1990) , searched for meaning in children's errors and misguided thinking. The emphasis on fallibility prominent in the heuristics approach characterises this work as part of the misconceptions movement.

Gradually the misconceptions approach has been subjected to criticisms in terms of its narrowness and lack of explanatory power. It is worth looking carefully at these criticisms, since I argue that the research in children's understanding of chance continues to be dominated by the attitudes which underpinned the misconceptions approach.

Perhaps the most eloquent critique of the misconceptions approach was that put forward by Smith, diSessa and Rochelle (1993). (but see also Chapter 2 in Noss & Hoyles, 1996) . The authors set out to examine and re-evaluate the misconceptions research. Rather than knowledge being generated through the replacement of previously flawed ways of interpreting the world, the authors argue that these flawed ideas contribute to the generation of new knowledge.

The authors draw on the corpus of misconceptions research to identify seven elements that underlie that body of work, though they do not claim that all seven apply to any individual researcher.

(i) Students have misconceptions.

(ii) Misconceptions originate in prior learning.

(iii) Misconceptions can be stable and widespread among students. They can be strongly held and resistant to change.

(iv) Misconceptions interfere with learning.

(v) Misconceptions must be replaced.

(vi) Instruction should confront misconceptions.

(vii) Research should identify misconceptions.

The researchers of the heuristics used by adults in making judgements of chance have tended to ignore explicit suggestions about how their work impacts upon teaching and learning though I have already discussed implications that can be inferred from this work.

Thus, although (ii), (v) and (vi) are only present in the heuristics work by inference, it is clear though that (i), (iii), (iv) and (vii) are quite explicitly present. According to the schedule of Smith, diSessa and Rochelle, the work of Kahneman and Tversky et al sits comfortably into this research methodology.

Smith, diSessa and Rochelle refute the misconceptions approach as lacking explanatory power in three areas:

(i) It offers no account of productive ideas that might serve as resources for learning. In particular, it only considers a narrow range of contexts in which the misconceptions occur.

(ii) Replacement is not an adequate model of learning. Evidence that concepts are more like clusters of related ideas suggests that it is unlikely such complex clusters could be simply replaced in a way that a unitary model of a concept might.

(iii) Confrontation is not an appropriate model of classroom learning. The rational replacement of one conception with another requires criteria for judgement but the confrontation/replacement models do not throw light upon how such criteria are generated.

The authors argue that novices possess intuitive knowledge which contains abstract elements. When novices are in situations which are close to their areas of competence, they are able to use abstract thinking, which is not bound to the surface features of the context. Novice/expert studies have however examined novices acting in areas in which they are not competent and have therefore not been able to observe their competencies.

The heuristics research is full of questions directed at people for whom those questions are clearly not close to their areas of competence, or in situations where appropriate tools to explore the questions are unavailable.

In discussing some examples drawn from their own studies, the authors conclude that novices in reasoning about the physical world:

(i) seek deeper explanations of the causality in situations than are immediately and superficially apparent,

(ii) attend extremely selectively to features of situations, ignoring many surface features to focus on what they see as causally relevant, and

(iii) apply principles.

They argue that:

> Novices appear to think concretely when they have been asked to classify problems that they are unable to solve and have nothing but generic, non-causal descriptions to rely on. Experts are classified as abstract because they have the particular abstractions selected as relevant — those that happen to solve the problems posed in research studies.
>
> (p. 131)

According to Smith, diSessa and Rochelle, it should be possible to find new learning environments which are closer to the pre-existing areas of competence of young children, in which case, it may be possible to observe their search for underlying principles. Such an approach may provide a richer evidential base for understanding the pedagogic implications of children's sense-making of the stochastic than an emphasis on how people fail when operating outside of their area of competence.

The move from novice to master is, according to Smith, diSessa and Rochelle, partly about shifts in the applicability of knowledge. The authors suggest three roles for prior knowledge in scientific expertise:

(i) To provide raw material for formulating scientific theory.

(ii) To support qualitative reasoning, and

(iii) To map everyday situations to theoretical representations.

They argue that:

> It seems more productive to study the roles that naive physical conceptions continue to play in expert reasoning than to suggest that the main issue in acquiring expertise is to remove and replace them.
>
> p. 145

The authors conclude by reconceptualising and reformulating the misconceptions approach.

(i) Casting misconceptions as mistakes is too narrow a view of their role in learning.

(ii) Misconceptions are faulty extensions of productive prior knowledge.

(iii) Misconceptions are not always resistant to change.

(iv) Replacing misconceptions is neither plausible nor always desirable.

(v) Instruction which confronts misconceptions is misguided and unlikely to succeed.

(vi) It is time to move beyond the identification of misconceptions.

This refutation of the misconceptions approach places great emphasis upon the intuitive knowledge that children and adults possess and across which even expert knowledge is apparently distributed. I therefore turn my attention now to studies on intuitions of randomness and probability.

2.4. INTUITIVE NOTIONS OF RANDOMNESS AND PROBABILITY

It has been suggested that the lack of consistent feedback from stochastic phenomena contributes to the underdevelopment of operational thinking. This line of thought suggests that concrete operations are missing causing an obstacle in the teaching/learning process and allowing the domination of unreliable intuitive thought (Borovnik & Peard, 1996):

> With probability, it has been noted, the concrete operations that form the basis of this process of concept acquisition are missing, thus hindering the reflection phase. The individual formation of concepts cannot be provoked by a hierarchical sequence of actions and reflections. This increases difficulties in any interaction between teachers and learners. Furthermore, the subjective domain of experience is more or less dominated by idiosyncratic and uncontrollable intuitive thought, as there is no direct feedback and control to real world experience.
>
> (p. 247)

In contrast, the view that pedagogic changes can enhance children's intuitions for the stochastic is supported by Fischbein (1975), who is much more optimistic than a strong inference of the heuristics research would predict . Fischbein is interested in the nature and role of intuitions as a vital form of knowledge in the process of working with ideas.

Intuitions have, according to Fischbein, some similarities to perceptions, in the way that they are compact, but they also have similarities to analytical knowledge in that they are derived knowledge and do not have direct links with the object as is the case with perceptions. Analytical knowledge is too remote and cumbersome to be useful in many modes of momentary thinking, whereas intuitions, being compact, are available to guide thinking and plans of action.

Fischbein reports how, when subjects were required to predict the outcomes of a repetitive series of stochastic trials, they were, even from an early age, able to tune the proportions of their predictions to the relative frequencies of the outcomes. It would seem therefore that even very young children have intuitions about relative frequencies. Fischbein proposes how such intuitions can be actively developed if the child operates in contexts which are structured in supportive ways (Fischbein, 1982).

> For instance, in order to create new correct probabilistic intuitions the learner must be actively involved in a process of performing change experiments, of guessing outcomes and evaluating chances, of confronting individual and mass results *a priori* calculated predictions, etc. New correct and powerful probabilistic intuitions cannot be produced by merely practising probabilistic formulae. The same holds for geometry and for every branch of mathematics.
>
> (p. 12)

Indeed, Fischbein argues that such contexts must be inherently attractive to the learner, who will need to recognise in the situation a reason for him or her to want to interact, to appropriate the knowledge (Fischbein, 1982).

> When considering the possible impact of a body of information or a pattern of mental procedures on the dynamics of productive thinking, we must take into account the kind and the strength of credibility attached to them by the learner.
>
> (p. 13)

There is a certain irony here. Piaget claims that probabilistic thinking is a very late development in the child's evolution of knowledge and Fischbein counters that initial intuitions would become established given appropriate experiences. In fact, children *are* particularly attracted to contexts where the laws of probability are central. Most children love games based on dice or playing cards, or computer games where unpredictable events happen constantly. They enjoy games and sports, which are amenable to probabilistic interpretations. Generally, children seem to enjoy coincidences and unexpected happenings.

Such stochastic experiences should be fundamentally important in children's development of intuitions about chance and probability. And yet, it does seem that probability is particularly difficult to learn.[5] This observation is consistent both with Piaget and Fischbein. The former would argue that it is inevitable since such young children have not developed formal operations that would allow them to construct a notion of proportion. Fischbein (Fischbein, 1975) sees such experiences as central ....

> Stochastic experience is to probability what spatial experience is to geometry.... The spatial image is the 'soul' of geometry, even though we may sometimes pretend to ignore it....Probabilistic intuitions also involve images — images of dice, coins, boxes, and so on, but these images have a mere auxiliary function. They are not intrinsic to the reasoning, in the way that spatial images are in geometry...The germ of intuitive reasoning about probability lies in natural 'experiments' with stochastic results, which involve predictions and random draws or other equivalent actions.
>
> (p. 16)

.... but Fischbein would claim that the experiences which children enjoy in their games and pastimes are not structured in such a way as to make the underlying probabilistic ideas explicit. As a result their intuitions do not in fact develop.

Even worse, Fischbein claims that school, with its emphasis on causality and determinism, has a counter-productive effect on the development of such pre-operational intuitions (Fischbein, 1975):

> This is why the intuition of chance remains outside of intellectual development, and does not benefit sufficiently from the development of operational schemas of thought, which instead are harnessed solely to the services of deductive reasoning.
>
> (p. 73)

Fischbein suggests that this withering of intuition about chance through a lack of nourishment in school provokes a need in the human mind to make the random more reasonable. As a result intuitions to do with probability develop a series of heuristics, which are often subject to bias.

Falk et al support this view quoting evidence that children as young as six years were able to perform at levels better than random when asked to optimise the chance of choosing a pre-determined outcome by selecting the most appropriate possibility space (Falk, Falk, & Levin, 1980). Like Fischbein, they infer that more attention should be placed in schools on the stochastic.

> One of the aims of teaching about probability in the first grades should be to restore the balance in favour of indetermination.
>
> (p. 202)

More recently, Fischbein has argued that some probabilistic intuitions are affected by tacit intellectual schemata, which strengthen with age, but are actually counter-productive in that they only apparently fit the problem (Fischbein & Schnarch, 1996). For example, we grow more inclined to employ the causality principle that the antecedent determines the consequent, leading to the ignoring of information where the association needs to be considered independently of a time base.

The authors suggest that schools might adopt a pedagogy in which children play games in order to experience randomness and build on this informal knowledge, though as I observed in earlier sections such approaches do not necessarily offer a very high chance that the children will attend to the mathematical structures within the game.

Falk also argues that schooling can bring about changes in heuristic thinking such as the equiprobability heuristic described earlier (Falk, 1992). By observing students' thinking about the notorious Three Prisoners' Problem, she argues that the uniformity notion is an example of a primary intuition (after Fischbein) and such intuitions are resistant to change, though she offers a range of secondary intuitions which are more easily learned through schooling. In particular, Falk claims that we must emphasise the protocols which determine how the random processes are generated.

> The lesson suggested by these studies is that if experimenters (and subjects) who study statistical intuitions highlight the explication of the (random) process that generated the data, some of the well-documented fallacies may disappear.
>
> (p. 219)

Indeed, she adds to the criticism of Kahneman and Tversky's methodology by arguing that their questions rarely drew attention to how the random phenomena were generated which might, according to Falk, explain the tendency to ignore base-rate frequencies.

Whereas, Piaget argues that the development of probabilistic thinking arises out of the human mind's need to find an operational way of thinking about random mixtures, Fischbein proposes that, on the contrary, intuitions about relative frequencies exist from a very early age but these are repressed by the school culture in favour of deterministic thinking, and, as a result, the mind substitutes a series of heuristics to help it rationalise stochastic events.

Though Fischbein's arguments are seductive, there remains a concern that the notion of intuition is too vague. There has been a plethora of attempts to define intuition, resulting in confusion rather than a specific and unambiguous description. It is therefore difficult to pin down the argument that conceptual change might be deeply contingent upon situational factors. In order to consider this issue further, I have to move the focus away from randomness, indeed away from mathematics. In part two, I will consider the role that the setting has on cognition more generally, and move towards a model of conceptual change which is able to accommodate the influence of setting and can be applied to the construction of mathematical and stochastic meaning.

PART TWO

2.5. SITUATED COGNITION

The work of Jean Lave has inspired a movement which has investigated the relationship between context and cognition (Lave, 1988). In her seminal work, Lave argues that conventional psychology has studied cognition in an artificial laboratory setting divorced from an everyday context. This approach has been consistent with a functional view in which culture is seen as a constant influence on the individual's cognition, a factor which can clearly be controlled without any loss of analysis. The echo with the heuristics research is loud and clear.

Consequently, she argues for a coming together of anthropology and psychology in which the unit of analysis is the whole person as he or she interacts within that individual's everyday settings. The school, for a child, is an important part of the everyday though the everyday encompasses all aspects of that person's routine activities.

In a powerful attack on research studies into transfer, she concludes by claiming that the notion of transfer is itself flawed. She argues that such research treats knowledge as an objective static reality and successful transfer as the ability of a researchee to find a preconceived (by the researcher) formulation of a solution to the problem. In transfer research, the problems are themselves not representative of everyday situations. For example, Lave points out that in everyday situations, jpf's (just plain folks) choose to solve problems. Furthermore the situation and the problem interact so that the problem is socially constructed. She argues that past transfer research has ignored the situation and the motivation that the problem solver had for engaging with the problem in the first place. Lave sees cognition as socially situated activity.

> The math observed appears to have a generative relation with the ongoing activities and at the same time to be shaped by them.
>
> (p. 68)

Lave suggests the following dimensions which cause activity to be constituted in situationally specific ways:

- If situations, occasions and activities are interrelated, these relations must shape arithmetic in practice.
- If a problem must be recognised in order to exist, it is not possible to locate problems exclusively either in settings or in cognitive processing — both are involved. Thus, individuals experience themselves as in control of their activities, interacting with the setting, generating problems in relation with the setting and controlling the problem-solving processes.
- The salience of any given activity when it unfolds in different settings varies. In the work place, the degree to which ongoing activity is organised in terms of mathematical concerns varies from being the main concern to (perhaps more often) of little relevance.

Lave argues that structuring resources exist in any situation and are fundamentally important in the development of knowledge. In a beautifully clear metaphor, Lave describes how she knits and reads at the same time but how one activity influences and is influenced by the other. So, she can read until she gets to the end of a row and then she needs to focus on the knitting for a while. The tightness of the knitting is influenced by the tenseness of the plot.

The central theme of her work is that, when we do not artificially remove setting from cognition, the structuring forces in that setting shape the constitution of that knowledge, which in turn influences the way that the person-acting interacts with those structuring forces. This idea is elegantly expressed in the following three quotations:

> The central idea is that 'the same' activity in different situations derives structuring from, and provides structuring resources for, other activities.
>
> (p. 122)
>
> Activity such as arithmetic problem solving does not take place in a vacuum, but rather, in a dialectical relationship with its setting.
>
> (p. 148)
>
> It was argued that what, in subjective terms, is the "same" activity ("arithmetic") takes different forms across situations and occasions, as it unfolds through the articulation of varied structuring resources in varying proportions. The discussion of money management provided demonstrations that arithmetic in practice is never merely that, but is the product and reflection of multiple relations — of value and belief, of people with each other, and of the conditions for producing and reproducing activity over time. Together they structure and are structured in activity
>
> (p. 174)

The relationship between knowledge and the situation is so deeply connected that, in its strongest form, the situated cognition movement claims that all knowledge is situated. Though many who have followed Lave's initial work would not state their claim in such strident terms, there is plenty of evidence supporting Lave's broader claims. I pick on one as further evidence of the relationship between situation and cognition.

Nunes studied the nature of informal mathematics (Nunes, Schliemann, & Carraher, 1993) by comparing the use of mathematics outside the classroom with that used in schools. Early experiments found large differences between the way Brazilian children carried out mathematical tasks when acting as street vendors and how they carried out similar[6] tasks in the classroom. Generally the tasks were completed more successfully in the informal context. It was unclear whether these differences could be attributed to the differing social conditions or the oral versus written nature of the tasks. Further experimentation led to the conclusion that the oral/written nature of the tasks brought about differences in the way that the problems were tackled even when the social conditions were similar.

Nunes concludes that street mathematics preserves meaning in a way that written arithmetic practice does not.

> Street mathematics is oral and preserves much of the meaning of the situations at hand. Mathematical practice in school is written and leaves out as much of the specifics of situations as possible in striving for generality.
>
> (p. 49)

The question now was whether preservation of meaning would result in loss of generality. One claim for schooling is that it teaches skills which can be applied in many different contexts. In fact the informal methods proved to be more generalisable than did the formal methods. Even when the problems were inverted so that the informal method had to be transformed into something which would never be carried out in the everyday contexts, the informal methods still proved to be superior.

Nunes draws the following powerful conclusion:

> Thus, representation of the particulars of the situation does not imply that the subject is restricted to understanding that situation. There is ample evidence for flexibility and generalizability of the pragmatic schemas of street mathematics. There is also plenty of evidence of the disconnection between people's knowledge of street and of school mathematics.
>
> (p. 147)

The strong interpretation of situated cognition presents a powerful attack on the very notion of mathematics, which sees itself as context-free. The whole notion of abstraction, drawing away from the context, so central to the prevailing view of mathematics, is brought into question.

This review of the situated cognition standpoint offers powerful evidence that there has been an enormous failure to consider the role of context in studying people's judgement of chance. In fact, the situated cognitionists have looked at a very narrow aspect of mathematics, principally arithmetic, which is often taught in very prescriptive fashion in our schools. In order to evaluate the implications of this research for mathematics more widely, we need a theory of conceptual change which can accommodate situated cognition and throw some light upon the question of how structuring forces may shape the construction of meanings for stochastic processes.

2.6. CONCEPTUAL CHANGE

Let us first review the development of the thesis so far. The argument for the refutation of the strong inference from the research on bias in the use of heuristics to make judgements of chance relies largely on the acceptance of the weaker claim that such intuitions are subject to environmental factors such as schooling. The acceptance of this weaker claim, though supported by the work of Fischbein and others, is hindered by a lack of clarity in any theoretical formulation that attempts to define the nature of an intuition. How can we determine what changes an intuition if we are unclear about what an intuition is? Indeed the same lack of clarity is apparent in most other attempts to describe concepts and conceptual change.

The situated cognition movement proposes that structuring forces shape (and are shaped by) cognition through a number of mechanisms which are quite clearly delineated. Even this approach though fails to offer an elaborated model of what it is that is changing. If I am to locate more precisely the position of *meaning*, I need a clear model of conceptual change.

The most compelling description that I have found arises from the study of students' construction of meanings for the Laws of Physics, in particular Newton's Laws of Motion. In his early work in this area (diSessa, 1982), diSessa found much evidence for an Aristotelian view of forces in an apparent expectation that things should move in the direction last pushed. There was a marked lack of influence by schooled physics. A growing assertion began to emerge in which current knowledge is fundamentally linked to subsequent development and that expert knowledge is encoded through distribution across many intuitive but structured pieces of knowledge. Taken to its logical conclusion, we might imagine a series of primitive notions which stand without significant explanatory substructure or justification, in a similar relationship to rest of an individual's knowledge to that between the axioms of mathematics and the rest of mathematical knowledge (diSessa, 1983).

> In axiomatic mathematics it has long been recognised that beneath all the complex of definitions and theorems must exist a special layer which serves as foundation for the rest......In science as well, though somewhat less prominently displayed, selecting the primitives of a theory is an important and complex process.
>
> (p. 15)

diSessa refers to these primitive notions as phenomenological primitives, p-prims for short. These p-prims are a collection of heterarchical and rich ways of seeing and sometimes explaining the world. Some are compatible with formal physics and so encouraged, thus taking on higher priority, others are not so. P-prims are not in themselves laws of physics but serve a variety of purposes such as heuristic cues to more specific technical analyses. Some p-prims lose status, being cut apart, explained in terms of higher priority ideas.

> P-prims are relatively minimal abstractions of simple common phenomena. Physics-naive students have a large collection of these in terms of which they see the world and to which they appeal as self-contained explanations for what they see. In the process of learning physics, some of these p-prims cease being primitive (and are seen as being explained by other notions), and some may even cease being recognised at all. But many become involved in expert thought in very particular ways.
>
> (p. 32)

This view of "knowledge in pieces" is in stark contrast with competing theories for the development of scientific knowledge. At the opposite extreme is what diSessa calls the theory theorists, who regard learners as having models of the world which are challenged by schooling and replaced by new models wholesale (diSessa, 1988).

In the specific context of probability, Konold argues that apparent conflict in the eyes of the expert may simply not exist for the child who can simultaneously hold supposedly inconsistent views (Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993).

> one way to produce conceptual change is to create situations for which answers based on a particular incorrect intuition produce cognitive conflict. .... The results of the present study suggest one limitation to the cognitive-conflict approach — a situation designed to contrast normative with informal reasoning may produce no conflict.
>
> (p. 412)

P-prims are a central tool in what diSessa calls "an intuitive sense of mechanism", (diSessa, 1993), a weakly organised system, involving many simple elements whose origins are unproblematic, as minimal abstractions of common events, and which is constrained by a lack of depth in justificatory structure and an inability to resolve conflicts within the system.

According to diSessa (1993) , this sense of mechanism has the capability to :

> Assess the likelihood of various events based on generalisations about what does and does not happen;
>
> Make predictions and "postdictions". That is, one can trace entailments forward or backward in time, explaining what will happen on the basis of what is the case, and explaining what must have been the case in order for the present circumstances to exist;
>
> Give causal descriptions and explanations. That is, one can look at a physical event and assign credit or blame for what happens to certain aspects of the circumstances and to general facts about the world.
>
> (p. 106)

P-prims are phenomenological in the sense that they are superficial interpretations of experienced reality. They are ready schemata in terms of which one sees and explains the world. They are primitive in that they are used as if they need no justification but also in that they are primitive elements of cognitive mechanisms.

P-prims are brought into activity according to the values of two priority systems. The way that a particular p-prim's transition to an active state is affected by other previously activated elements is called *cueing priority.* A high cueing priority means that only a small additional stimulus is needed for it to be activated. Suppression can be represented as negative cueing priority.

Once a p-prim has been activated, further processing involves the *reliability priority* which affects that element's state at future times. Reliability priority describes potential feedback that can reinforce or undo the initial activation.

The above description summarises a highly elaborated model of the internal resources which we bring to bear during the process of sense-making. This model is extended to consider how these structures then change and develop.

The large but unstructured set of p-prims becomes tuned towards use in instructed physics. Whilst they remain unstructured, structuring is only local with priorities only established in small neighbourhoods. In these circumstances, there may be no way to resolve conflict. During "tuning towards expertise", the priority of some p-prims becomes enhanced or reduced. As a result, there is a subtle but highly significant revision in the function of p-prims.

> Undoubtedly some entirely new p-prims are generated as the learner's descriptive apparatus changes to focus on different features and configurations in the physical world. But, a more drastic revision in the intuitive knowledge system is in the change in function of p-prims. They can no longer be self-explanatory but must defer to much more complex knowledge structures, such as physics laws, for justification. P-prims come to serve weaker roles, as heuristic cues to more formal knowledge structures, or they serve as analyses that do their work only in contexts that are much more particular than the range of application of the general or universal laws of physics.
>
> (p. 114)

diSessa argues that the role of p-prims changes from a shallow explanatory system to the gradual clustering of p-prims into distributed encoding, the re-use of knowledge elements as parts of expertise such as aspects of sanctioned physical ideas.

In a recent paper (diSessa & Sherin, in press), the authors refer to two strategies, called readout strategies, for gaining information from phenomena: integration — the collection, selection or combination of diverse "observations" to determine what we wish to "see", and invariance — how observations in different circumstances can manage to determine the same "information".

The general class of knowledge and reasoning strategies that determines when and how some observations are related to the information at issue is called the causal net.

> The relations between readout strategies and the causal net are intimate. One looks for things which are related (via the causal net) in order to determine some quantity. Indeed, even seeing those secondary features may involve additional inferences for the same or another causal net. In general, readout strategies and the causal net should co-evolve as learning occurs. There should be episodes of "conceptual bootstrapping", where causal assumptions drive the learning of new readout strategies. On other occasions, "noticings" may drive reformulations in the causal net. In general, characteristics of one will have important influences on how the other behaves and develops.
>
> (p.19 of draft version, dated July 10th, 1997)

The process of change can be elaborated as the changing of old readout strategies or the development of a completely new readout strategy and as the refinement of an old causal net or the invention of a new causal net.

P-prims constitute the causal net for beginners' use in bootstrapping their way towards understanding physical concepts. The authors propose that integration and invariance are difficult precisely because of the lack of system and the rich variety of p-prims that make up the causal net.

Readout strategies and causal nets combine to make up what the authors call co-ordination classes, which are:

> systematically connected ways of getting information from the world .... the prototypical task for co-ordination classes is getting information. This contrasts with the prototypical task for category-like classes, which is to determine whether something is or is not a member of the category.
>
> (p.14 of draft version, dated July 10th, 1977)

I find here resonance between some of the literature on intuitions of the stochastic and this model of conceptual change. For example, Konold refers to the inconsistencies in students' thinking about the stochastic (Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993):

> ... the typical person has knowledge about a variety of uncertain situations, but that knowledge is incomplete and not integrated. Different situations access different pieces of this knowledge. Thus in one problem, a person may appear to reason correctly, but in another, this same person may reason in ways that are at variance with probabilistic and statistical theory.
>
> (p. 393)

In the same paper, Konold refers to students who offered explanations of the behaviour of a coin as self-evident, and as tools for explaining other aspects of stochastic behaviour.

The picture of conceptual change that emerges from diSessa's work is more precise than that offered by Fischbein. We now see intuitions in terms of bunches of small, self-explanatory primitives, abstracted directly from experience, but structured, by enhancement or reduction of priorities through interaction with phenomena and social forces such as schooling. The primitive notions are in a sense in competition in that several different p-prims can be activated in any particular setting. Aspects of the setting cue particular p-prims according to the relative priority values. It is therefore possible, especially prior to the structuring that experience may bring, that slightly different circumstances could activate contradictory p-prims.

We must, though, consider whether this theory, so well elaborated but built on experimental evidence of students learning physics, has relevance to a mathematical domain. Let us consider this question in relation to abstraction, an idea which is central to the discipline of mathematics.

2.7. ABSTRACTION

What does diSessa's theory have to say about abstraction? His notions of integration and invariance are helpful in this respect. These read-out strategies aim to collect information within and across situations and are abstracted away from the phenomenon through the activation of p-prims, which are themselves already simple abstractions of reality. Integration and invariance may bring about some restructuring in the form of changes in the values of those priorities.

Though mathematics may often be concerned with elements which are not in themselves phenomenological, it is not too difficult to envisage the construction of meanings for mathematical elements to be in terms of p-prims which had their roots in the phenomenology. In many cases of mathematical learning, the journey back to those phenomenological roots would be far too complex and convoluted, but in the case of young children's understanding of chance, we may find relatively straightforward connections.

Abstraction from a situation to mathematics has classically been presented as one of removing the context, a translation from action to cognition. Mathematics itself is seen as possessing ever ascending degrees of abstraction, each carefully built on lower, less abstract pieces of mathematics. It is certainly true that formal mathematics is rigorously defined in this fashion, though it is not clear that we should necessarily ascribe features of the discipline to the process of learning about that discipline. The former belongs to the subject of epistemology, whilst the latter belongs to psychology. We have seen the powerful role of context in the learning process, almost by definition eliminated from mathematics as a formal discipline, and so we should not assume that the notion of abstraction applies equally to both.

Sensing this increasingly apparent contradiction, Turkle and Papert (1991) have called for a re-evaluation of the abstract and its flipside, the concrete:

> Observation of the soft approach to programming ... provides examples of the validity and power of concrete thinking in situations that are traditionally assumed to demand the abstract. It supports a perspective which encourages looking for psychological and intellectual developments within rather than beyond the concrete and suggests the need for closer investigation of the diversity of ways in which the mind can use objects rather than rules of logic to think with.
>
> (p. 166)

Wilensky has responded to the appeal from Turkle and Papert by proposing that the concreteness of a concept lies in the relationship between the learner and the concept (Wilensky, 1991). He suggests that as we become more and more familiar with an idea, we make increasing connections with it; we concretise the concept.

In this view, ideas become increasingly concrete. Advanced mathematical concepts are neither more nor less abstract per se. For an individual who has not had the opportunity or does not yet possess the internal resources, a concept for them will be abstract, disconnected. That same concept may be quite concrete for another individual.

Noss and Hoyles summarise the position of abstraction in the light of these developments (Noss & Hoyles, 1996)

> Thus the hierarchy begins to look a little shaky. The challenge for the individual learner becomes one of constructing multi-faceted connections between activities and experiences that are 'in some way' similar. Abstraction becomes a problem of how to add new friends and relations, not to ascend to unattainable heights. And from this perspective flows the educational corollary that it might be possible to design educational environments in which the process of abstracting becomes part of the lived-in culture of experience.
>
> (p. 47)

We learn from diSessa's model of p-prims that, for the uninducted, meaning can only be rooted in experience. And yet there is here an alternative sense for abstracting. As the child interacts within a situation, structuring forces (external) and p-prims (internal) come together and may result in restructuring of those p-prims. If such changes occur, then the newly structured p-prims will possess attributes not apparent prior to structuring. For example, whereas the original form appeared to be self-explanatory, the p-prims may now have explanatory power, perhaps by connection with other p-prims. Furthermore, whereas the original p-prims may have been in contradiction with other p-prims which were activated under nearly similar circumstances, there will now be less evidence of such contradiction.

It is here that we can locate meaning, in the forging of new robust, explanatory and consistent connections We might predict from the situated cognition movement that such meanings will be deeply contingent on the situation. Hoyles and Noss encapsulate this dependence between abstraction and situation in their term *situated abstraction* (Hoyles & Noss, 1993). These situated abstractions are seen as constituent material within a broader view of conceptual change in computer-based settings.

Noss & Hoyles propose the term, *web*, to describe the union of those structures built into the software by the programmer and those mental structures *forged* and *re-forged* during the activity by the learner (Noss & Hoyles, 1996).

> • it is under the learner's control;
>
> • it is available to signal possible user paths rather than point towards a unique, directed goal;
>
> • the structure of local support available at any time is a product of the learner's current understandings as well as the understandings built by others into it;
>
> • the global support structure understood by the user at any time emerges from connections which are forged *in use* by the user.
>
> (p. 108)

Having located meaning, it is a simple feat of the imagination to conjure up an image of a domain, possessing the structuring resources aimed at optimising the process by which p-prims are structured or meanings are constructed (though it is far from trivial to imagine the specific attributes of any such domain). A domain for stochastic abstraction is an environment, tuned for the construction of meaning, by placing at the forefront of its design, consideration of (i) the internal resources, which a child is likely to bring to activity within the environment, and (ii) the structuring resources, which aim to optimise the construction of reliable and consistent situated abstractions of the stochastic.

As a first attempt at sketching out the nature and potential of a domain of stochastic abstraction, I discuss in the next section the specific pedagogic contribution made by the development of microworlds. We can regard such microworlds as prototypical domains of abstraction. Though these prototypes have tended to focus on generic problem-solving skills, and on topics from mathematics and science other than the stochastic, we can nevertheless draw on this earlier work to visualise the outline of what such a domain for stochastic abstraction might look like.

PART THREE

2.8. THE PEDAGOGIC CONTRIBUTION OF MICROWORLDS

Examples of the careful design of domains of abstraction began to emerge in the 1960's when a team, headed by Papert and Feurzeig, was developing the computer language, Logo, at MIT. The early work was by necessity non-graphical in nature, but technological advances enabled the addition of the first domain of abstraction, the turtle graphics microworld (See Papert, Watt, diSessa, & Weir, 1979; Watt, 1979). That early research had four explicitly stated themes:

(i) Can elementary school students learn to program?

(ii) What is the relationship between programming and maths?

(iii) What is the relationship between programming and cognitive style?

(iv) How can we evaluate the work?

This pioneering work offered a new way of thinking about the learning of mathematics, an approach which proposed using Logo as a model for learning about problem solving and problem posing. In particular, they advanced the radical notion that children need to play with and use mathematical concepts within a supportive computer-based environment before being introduced to formal work with those concepts (Papert, 1972).

> When mathematizing familiar processes is a fluent, natural and enjoyable activity, then is the time to talk about mathematizing mathematical structures, as in a good pure course on modern algebra.
>
> (p. 18)

These initial ideas reached a climax (Papert, 1972) in which a radical vision of education was proposed. Since then, the work has been elaborated to the point where a new paradigm for the teaching and learning of mathematics, the *constructionist* approach, has been put forward (Harel & Papert, 1991).

The original four themes have not entirely disappeared in the subsequent decades, though there has been a change in emphasis from the focus on generic problem solving skills to the design of environments or microworlds[7], which focus on more specific aspects of mathematics without losing the pioneering spirit of that early work. Some generic issues have emerged from this developmental work which can be seen as informing the future design of microworlds.

2.8.1. Quasi-Concrete Objects

Turkle and Papert refer to the way that the computer offers access to formal ideas in a concrete way, since abstract mathematical ideas, represented in iconic form on the screen, can be manipulated directly by the user (Turkle & Papert, 1991).

> The computer stands betwixt and between the world of formal systems and physical things; it has the ability to make the abstract concrete. In the simplest case, an object moving on a computer screen might be defined by the most formal of rules and so be like a construct in pure mathematics; but at the same time it is visible, almost tangible, and allows a sense of direct manipulation that only the encultured mathematician can feel in traditional formal systems.
>
> (p. 162)

This feature of microworld design more than any other enables us to appreciate Papert's assertion that the concrete and the formal should be seen as styles rather than as hierarchical, in which the aspiration is always to move from the concrete to the formal or abstract. The image of the quasi-concrete object opens up the possibility of children manipulating and using mathematical objects, which conventionally would be seen as abstract but which are represented as if concrete on the screen. We begin to appreciate why Wilensky emphasises concretion rather than abstraction because, as children interact with quasi-concrete objects, they develop a relationship with the object. The familiarisation of the object is a fundamental part of the construction of meaning.

2.8.2. Using Before Knowing

Interaction with quasi-concrete objects allows children to separate features and behaviours of the object, a process which Hoyles and Noss refer to as *discrimination*, a first stage in their computer-mediated model of the acquisition of concepts (Hoyles & Noss, 1987). Their UDGS *(Using, Discriminating, Generalising, Synthesising)* model proposes that the child uses a concept prior to an understanding of the underlying mathematical relationships. In fact, such an approach relies upon the way in which the computer can validate the child's use of the concept on her behalf and offer helpful feedback. The child discriminates the salient features of the concept as he or she uses it. Gradually the child generalises the range over which the concept is applied until able to synthesise into an integrated whole this use of the tool with other representations of the same tool from other domains. Noss and Hoyles explored how children developed understanding of variable through the way in which the computer allows functional activity and formalisation to take place simultaneously (Noss & Hoyles, 1988). The pedagogical challenge becomes one of identifying the situated abstractions and of gaining some grasp of how it can be that, when meaning is constituted within situations, children can make connections across situations.

In our everyday lives, we typically use artefacts for particular purposes. Through that use, we learn about the effectiveness of the tool, its limitations, how well it serves that purpose and sometimes we may gain some understanding of how it works. This seems to be the natural way of learning. Mathematics uniquely has always been represented as different. In schools, mathematics is a subject where you learn how to generate the object before you use it. In practice, more often than not, the former task proves too difficult, especially when disconnected from purpose, and so we never reach the second stage of using the piece of mathematics. This is the likely story behind the research of Nunes, which showed that formal methods of instruction reduced meaning and therefore effective learning.

The computer offers the possibility of turning the learning of mathematics round so that using precedes generation, thus bringing mathematics more into line with natural ways of learning. Papert has recently referred to these ideas as the *Power Principle* (Papert, 1996).

> The principle is called the power principle or "what comes first, using it or 'getting it'?" The natural mode of acquiring most knowledge is through use leading to progressively deepening understanding. Only in school .... is this order systematically inverted. The power principle re-inverts the inversion.
>
> (p. 98)

2.8.3. Integrating the Informal and the Formal

diSessa suggests that experience is a way of engaging naive knowledge. Computers provide a medium for designing activities that build and integrate pieces of knowledge. A microworld may be able to integrate these fragments of knowledge by offering opportunities for their use, enabling the construction of meaning. At the same time, diSessa has suggested that we incorporate versions of the formal representations of the mathematical objects in such a way that the child may be able to make connections between the various formalisations and their informal use (diSessa, 1988).

> Building analytic or other formal tools right into experiential environments should become more and more a standard part of microworld design.....The idea is not to juxtapose experiential and formal points of view as above but to fuse them.
>
> (p. 64)

Papert (1982) has referred to the process of embedding powerful mathematical ideas into a microworld, as planting "nuggets of knowledge" which the learner encounters during informal activity. Technological developments allow us to express this notion in new and powerful ways. In particular, innovations in interface design enable us to express formal mathematics in both symbolic (e.g. programming) and iconic (e.g. direct mouse-oriented manipulation) forms. Indeed new learning opportunities may be possible in the connection of the symbolic and the iconic.

Formal representations of pieces of mathematics can be seen as either constructive or instructive (O'Reilly, Pratt, & Winbourne, 1997); the former is offered as a means for the learner to build new representations whilst the latter is offered as a finished article, an expression of culturally sanctioned mathematics. The microworld approach would place more emphasis on constructive representations than conventional pedagogies. Indeed the notion of connecting the informal and the formal may lie in the process of building with constructive representations.

2.8.4. Dynamic Expression

When Papert proposed the turtle as a tool for constructing a dynamic notion of angle (and of course much else), he acknowledged that the computer offers a medium which unlike paper and pencil can incorporate dynamic representations of the world. He suggests that the use of systems which are expressive of dynamic and interactive aspects of the world are more engaging to learn than static and abstract formalisms.

A critical aspect of microworld design is that the child is able to interact and control that dynamism. The control may at times be exerted by direct manipulation which has the advantage of immediacy. Control can also be exerted through programming. The act of expressing fuzzy ideas in a formal, conventional and rigorous language makes those ideas become more explicit. The ideas also take a form which can be observed and used by others.

It is a fundamental part of mathematics that ideas are expressed in this sort of way and so programming becomes akin to doing mathematics. Teaching the machine becomes a central microworld activity.

diSessa has gone as far as proposing that microworlds offer a new form of literacy (diSessa, 1995).

> ...by extending linear language into multiply connected, dynamic, richly textured graphical and interactive forms allowed by computers we may fundamentally extend the material bases for thinking and learning, and with them the whole practice of education.
>
> (p. 2)

2.8.5. Purpose and Utility

In conventional instructionist pedagogies, where operations precede use, meaning is lost. The microworld can encourage purposeful activity through the building and modification of artefacts. In so doing, situated abstractions are imbued with *utility* (Pratt & Ainley, 1997), in which the abstractions are seen as useful and the limitations of those abstractions are gradually discriminated.

In conventional approaches to instruction, purpose and utility are often confused. Consider a lesson where the central mathematical concept is mean average. Conventionally the lesson will be presented as being about average. The aim for both the teacher and the child will be to generate average. The child will learn something like 'add up the numbers and divide by how many there are'. That is the purpose. For the child, average has no utility; it is not distinguished from the purpose of the activity. In contrast, a microworld can present an environment in which average is used. If the microworld is well-designed, the child may learn that average is a useful way of comparing two sets when they are unequal in size (standardising). Or perhaps utility will be seen in terms of the average providing a better estimate than any of the individual experimental trials (estimating). Because the microworld can provide a quasi-concrete object called average, these utilities may be discriminated without even knowing how to generate average. The advantage may be that such an approach separates purpose and utility, as is usually the case in our everyday learning but in contrast to standard mathematics classroom practice.

2.9. TECHNOLOGY AND PROBABILITY

Whilst the discussion on technology has stretched across a variety of mathematical and scientific domains, I have only so far considered the contribution of the *microworld* approach.

There has in fact been considerable work on the use of technology in the teaching and learning of statistics. Much of this work is not closely related to probability, and so can safely be ignored by this review. (Indeed one branch of technological developments, that of Exploratory Data Analysis, aims to offer a pedagogical approach to statistical inference which avoids the subject of probability, partly because of its perceived difficulty[8].)

I propose to consider previous research on the use of technology in the teaching and learning of probability in two sub-sections. The first summarises research which has focused on the computer's ability to generate conventional representations of the stochastic. The second sub-section examines one contribution to the literature, which has brought the research on microworlds into the domain of probability and has therefore been particularly influential as far as this study is concerned.

2.9.1. Conventional Technological Approaches to Probability

Conventional uses of technology have adopted a black-box approach to the teaching and learning of probability. The method typically expects the learner to execute various commands to simulate experiments and account for the resulting graphical images. The findings from this work have often pointed to its limitations as far as learning is concerned. Nevertheless these limitations serve to point the way in which we might better design and use stochastic software.

- *Clarity in the model of randomness*

  There has been some speculation that the use of computers in stochastic work will be hindered by learners' concerns about the nature of computer-based randomness. It has been recommended that the pseudo-random nature of randomness on the computer may need to be made transparent as part of the activity (Borovnik & Peard, 1996)

  > It has been suggested that "the philosophical and possible intuitive troubles in the mind of individuals may hinder an effective use of computers if this aspect is not clarified.
  >
  > (p. 262)

  The algorithm used to generate pseudo-random numbers may be hidden. Though the details of these complex calculations are no doubt of little significance to most learners, it appears that a top-level of clarification may be desirable. We might conjecture that a microworld approach could achieve this clarification by encouraging the child to engage directly with the model itself.

- *Data is not forceful*

  Konold has described an experiment in which he placed bets against a student on the outcomes of a series of coin tosses (Konold, 1995). The account is amusing because it turns out that Konold himself was using an incorrect mental model of the situation, resulting in various monetary gains by the student. The power of the story lies in how reluctant Konold was to accept the force of the data. According to Konold, our beliefs about how stochastic phenomena behave are resistant to change. We are more likely to find stories and explanations for the vicissitudes of the data than to believe an alternative interpretation of the data.

We might conjecture that a microworld approach could use interaction with the theoretical model AND the data as a means of the child constructing more powerful meanings for the data.

- *Attention is a limited resource*

Konold has described how the technology itself was not necessarily engaging. He has suggested that the choice of task is fundamentally important in focusing the learner's attention. Konold (1994) outlined three features of a task which help to involve the student in the problem:

(i) the results should be counter-intuitive;

(ii) there should be multiple options for further analysis;

(iii) the task should model a real situation.

Konold's analysis suggest that attention must be paid to the design of tasks in which computer-based tools are utilised, as well as the design of those tools themselves.

- *Collection of Enough data*

Konold has described how there was a tendency for students to underestimate just how much data was needed to draw reasonably sound conclusions (Konold, 1995). It is usually far easier to collect data in a computer simulation than in a real experiment. Nevertheless, it may not be transparent to the learner just how much data is needed.

- *Variability is typically ignored*

Konold has described how simulations tend to focus on relative probabilities and to ignore variation. The task should encourage the learner to consider how for example proportions vary from experiment to experiment through repetition. The possibility of repeating experiments is a potential advantage of the use of technology but one which is rarely exploited to study variation.

- *The focus should be on sense making*

  Konold has argued that simulations offer us a way of testing our theories, not replacing them, and that theories should remain the primary focus (Konold, 1995). He argued that:

  > My own belief is that this approach has a chance of leaving untouched the informal notions students bring into the classroom. The approach I have used is to encourage students to articulate their informal theories, to make predictions from them, and to use the results of simulation to motivate alternative explanations.
  >
  > (p. 209)

  Though I would argue with Konold's use of the word *theory* here, which seems to me to attach too much structure to these loose intuitive ideas, we can conjecture that a microworld approach may be effective insofar as it can put the child's articulation of intuitive ideas at its heart.

This section has summarised the most relevant findings from research which has studied the conventional use of computers in the teaching and learning of probability. The limitations point directly at a number of pedagogic and cognitive barriers.

The use of technology can easily seduce one to believe that these barriers can be avoided. For example, the fact that computer simulations can facilitate the collection of data does not in itself suggest to the learner that large amounts of data are necessary. Similarly the potential to repeat simulated experiments does not necessarily mean that the child will identify the need to repeat an experiment. The microworld approach may be better placed to address directly these pedagogic and cognitive issues and to promote the structuring of intuitions (in the sense of the diSessa model of conceptual change). Indeed, there has been one major piece of work which has used the microworld approach to study students' conceptions of probability, and I wish now to focus on this influential work..

2.9.2. The Microworld Approach to Probability

In contrast to these conventional uses of technology, an alternative approach has been proposed (Wilensky, 1995a):

> .... instead of learning probability through solving decontextualised combinatoric formulae or being consumers of someone else's black box simulations, learners can participate in constructionist activities — they design and build with probability.
>
> (p. 152)

Uri Wilensky studied university students' conceptions of probability as part of wider research, touching topics such as proof, recursion, fractions as well as probability, and investigating the obstacles to learning (Wilensky, 1993). His work formed part of the Connected Mathematics project, which proffers a view of teaching and learning, closely associated with the constructionist paradigm, incorporating:

- a focus on the epistemology of mathematics;
- the notion that mathematical ideas are multiply represented;
- a collection of problems, puzzles, or paradoxes which highlight the key ideas and distinctions in a mathematical domain;
- mathematical ideas linked to other mathematical ideas and related cultural ideas.
- an enquiry-rich environment which promotes user designed constructions and conjectures.

Within this framework, one aspect of Wilensky's work concentrated on conceptions of probability, a domain identified as particularly rich precisely because of its seemingly inherent paradoxes and contradictions.

He used a specially adapted version of StarLogo, software originally designed by Mitchel Resnick, to study emergent phenomena — phenomena that are the result of the interaction of many locally interacting agents (Resnick, 1991). According to Resnick, people have difficulty in going beyond a "centralised mindset" in which they are constrained to view the phenomena as consequences of a single control, a sort of deterministic outlook, rather like Konold's notion of an outcome approach.

Wilensky asked some specific epistemologically oriented questions about conceptions of probability (Wilensky, 1993).

> Are the difficulties in understanding emergent phenomena due to their probabilistic character? Is there such a thing as a deterministic mindset by analogy with a centralized mindset? Is the change that transpires when we say someone understands probability a matter of incremental knowledge — a mere mastering of subject matter in a new mathematical area? Or is the change more fundamental — a global change in the entire way of looking at the world?
>
> (p. 100)

Interviewees were asked to work on paradoxical problems, often using StarLogo to formulate and test their ideas. It is clear from these extended interviews, that the students often floundered though, according to Wilensky, this struggle was associated with deep epistemological issues.

> In the probability study, we have witnessed learners blocked from developing their probabilistic knowledge by concerns about the epistemological status of concepts such as probability, random, and distribution.
>
> (p. 182)

Despite, or Wilensky would claim *because of*, these difficulties, Wilensky described how the students came to understand the ambiguity of statistics in everyday life, constructing and interpreting such statistics, finding connections, and so gaining some control over their everyday lives. Wilensky argued that they came to see randomness in a connected way, neither representing complete ignorance, nor just a mathematical formalism.

In particular, Wilensky has suggested that these connections helped his students to build bridges between stochastic and deterministic thinking.

> Typically, people dichotomize, seeing phenomena as "wholly random" .... or as deterministic. .... The kinds of constructions made by the interviewees, the negotiation of meaning for randomness, probability and distributions, are the kinds of bridges necessary to a less dichotomized view.
>
> (p. 184)

The struggle that students experienced to make these sorts of connections are summarised by Wilensky as a series of obstacles to probabilistic thinking. The list below is a précis of Wilensky's own summary.

- To think about probabilities, we often need to reason backwards in time, not a natural process in our everyday lives;
- The assimilation of probability into our culture is still in its infancy;
- The presentation of statistical data is often detached from its meaning and its

method of collection, preventing us from operationalising that data in order to make it meaningful;

- Short term memory limitations force the summarising of data, but these summaries can not be decompressed back to the raw data, with a resultant loss of meaning;
- We suffer in our everyday lives from a lack of feedback which could measure the adequacy of our probability judgements;
- To see our lives as experiments in probability requires us to make connections between our current situation and a large collection of similar situations in the past or potentially in the future, a massive mental construction.
- Similarly, the task of thinking about multiply interacting agents is simply too demanding and so we construct ways of thinking about average behaviour instead.

Several of these obstacles can be attributed to the inadequacy of short term memory. In this respect, Wilensky sees the computer as an extension to our memory systems which can help us to overcome many of these obstacles (Wilensky, 1997):

> A central conjecture of the Connected Probability project is that computational technology should do for probability what Hindu-Arabic notation has done for arithmetic. Computational environments allow large bodies of data to be visualised at once in a small space and large numbers of repetitions to happen in a short time. As a result, short term memory resource limitations can be overcome. The focus can then be on probabilistic reasoning.
>
> (p. 179)

Wilensky claims that, in using StarLogo, the students constructed new intuitions of the stochastic (Wilensky, 1997):

> The computer serves as a setting and symbolising medium for building mathematical intuitions
>
> (p. 180)

A central notion in Wilensky's use of StarLogo is that, unlike the more conventional uses of computers, the child interacts with the formalisms themselves to build new products, a process which brings the learner into a closer intimacy with fundamental epistemological and conceptual barriers embedded in the stochastic. Of course, the level at which this interaction is pitched is a critically important design issue (Wilensky, 1995b) ....

> The challenge for such an approach is to design the right middle level of primitives so that they are neither (a) too low-level, so that the extensible model becomes identical to its underlying modelling language, nor (b) too high-level, so that the application turns into an exercise of running a small set of pre-conceived experiments.
>
> (p. 154)

.... a design issue which will lie at the heart of one strand of this study.

2.10. SUMMARY OF CHAPTER TWO AND EMERGING FOCUS FOR THIS STUDY

We learn, through the work of Fischbein and diSessa in particular, that intuitions are many in number and that their informality, whilst providing flexibility and immediacy, at the same time allows inconsistency , incompleteness and lack of rigour. Intuitions are not aspects of a theoretical model held to be true until replaced by an alternative theory. In fact, they are sense-making devices, which enable us to interface with phenomena in meaningful ways. Intuitions are not static but constantly change in the light of experience, cued by aspects of the setting and shaped by the experience of using those intuitions. This image of multitudinous dynamic intuitions is one of complexity, and is the object of my study.

The review of the literature has enabled me to situate the construction of meanings for the stochastic in a grey area lying at the intersection of several domains of knowledge.

Science and mathematics

The stochastic itself lies in a fringe within which the disciplines of mathematics and science disciplines overlap. More than many aspects of mathematics, the stochastic has clear and immediate relevance to phenomenological experience. Such phenomena usually form the basis of scientific study and yet the stochastic is peculiarly resistant to study through conventional scientific method. The formalisation of the stochastic has brought it into the mathematical domain.

Informal and formal mathematics

As we go about our everyday activity, stochastic phenomena play an important part in our lives. We are required to make informal judgements of chance even though we rarely possess the intellectual tools that would enable us to make such judgements with great accuracy. In contrast, those who have been inducted into mathematical discourse possess exactly those tools. We can conjecture that the construction of meanings for the stochastic lies somewhere in the connecting of informal and formal views of stochastic knowledge.

Intuitions and operations

The informal and formal expressions of stochastic knowledge are paralleled by intuitive and operational models for the construction of that knowledge. These models take different slices through this complex domain and, in so doing, present quite different images of how such knowledge is constructed. Nevertheless we learn from Piaget's operational approach that meaning for the stochastic emerges through accommodation out of a rejection of the deterministic. Intuitive approaches suggest though that the heuristics that we develop to cope with everyday judgements of chance are inadequate.

A synthesis of the diSessa model of p-prims and the Noss/Hoyles model of webbing offers a picture of conceptual change in which primitive intuitions of randomness abstracted directly from experience are restructured through webbing with external tools. The construction of new meanings may take the form of situated abstractions, which we can also envisage as partially re-structured p-prims.

Situated cognition and decontextualised mathematics

The construction of meanings for the stochastic can also be seen as lying in the intersection of two extreme versions of the role of setting. At one extreme we see a movement which, in its strongest version, proposes that all knowledge is situated whilst at the other extreme is a view of mathematics as decontextualised. We can see the construction of meanings for the stochastic as demanding the building of connections between the pieces of mathematical knowledge intentionally planted into the arena and the phenomenological and social forces embedded in a setting.

The stochastic therefore represents a particularly rich domain in which to study children's intuitions. In the fringe between the various dimensions described above, we might expect to discover how mathematical meanings emerge from the interaction of such intuitions. Chance has been appropriated by mathematicians and, in so doing, randomness has become formalised, subject to rigorous analysis, respectable in mathematical terms. Yet the roots of randomness remain identifiable in everyday culture and activity, offering opportunities to study the interplay between formal mathematical expressions of the stochastic and everyday informal intuitions of randomness.

Part three of this review of the literature has given an overview of developments in the design of microworlds and a first glimpse of what a domain of stochastic abstraction might look like. We have also gained some initial ideas of how technology can contribute towards the teaching and learning of probability and in particular to the construction of new meanings, at either a conceptual or an epistemological level, for the stochastic. By observing children using, modifying and building new representations in the form of computer-based instantiations of stochastic phenomena, I aspire to throw light upon the grey area that lies in the intersection of these different domains of knowledge.

Certainly the review of the literature throws up some important gaps in the body of knowledge:

- There is little research on *young* children's construction of meanings for the stochastic. Much of the work discussed above was carried out with students at university and adults. Some work with very young children has been carried out but little with children in the 9-12 age range[9]. In fact, I regard this as a critical age since it is often around this age or just after that formal approaches to the study of chance are introduced.

- There is a need to pin down what sorts of intuitions such young children are likely to possess as a result of their everyday activity. At the same time, we must ask what is it that experts understand that young children do not and what sorts of pedagogic tools might begin the shaping of naive knowledge towards the expert point of view, if indeed such a process is at all possible.

- Much of the study of stochastic intuitions has been atheoretical (Fischbein is one notable exception). At the same time these studies have often taken the form of snapshots of how people seem to make sense of a stochastic situation at a particular point in time. We may gain much insight by examining how stochastic intuitions change and develop when they come into juxtaposition with external tools. We have seen how Wilensky has worked with adults in this sort of way but, by doing so, he is inevitably studying the interaction of schooled (secondary in Fischbein's terminology) intuitions with situated tools. Though this work is extremely important and informative, I wish to study children whose intuitions are principally based on everyday abstractions of the behaviour of stochastic phenomena.

- The theoretical development on conceptual change that we owe to diSessa has been based on observations of students trying to make sense of physics concepts, such as force and velocity. Though this model has been shown to be powerful because of the way that it incorporates the influence of setting on an individual's primitive intuitions, there has been no explicit attempt to test this model out in a context where the concepts are regarded as more mathematical than scientific. The domain of the stochastic, lying in the intersection of science and mathematics, may be exactly the right place to make a first attempt at this task.

- Similarly, Noss & Hoyles have built a connectionist model of webbing, based on their extensive research on children using Logo microworlds. We should gain a better understanding of the webbing construct by using webbing as a framework for the study of children's construction of stochastic meaning. In particular there is a need to develop the notion of webbing to enable us to have a clearer image of what it is that is being webbed and exactly how webbing operates. In particular, there is a need to identify how the specificities of the medium, the domain of stochastic abstraction, shape this webbing process. One broad aim for this study will be to identify what these generic features look like in the case of stochastic abstraction. I aspire to identify the specific features which might optimise the construction of meanings for stochastic processes.

This list of gaps in our understanding of conceptions of the stochastic can be re-formulated as a set of specific aims for this study. These aims will be developed and articulated in more detail in the next chapter.

NOTES

2 My use of the phrase *stochastic phenomena* is to be interpreted as phenomena which may potentially be be treated as stochastic by the learner, and is not meant to suggest that the randomness is somehow built into the phenomenon itself.

3 This belief seems to underpin the reason why the following well-known poster is funny: "Probabilities are all 50%. Something either will happen or it won't".

4 I find myself dissatisfied with that research in mathematics education which seems to lose sight of the central domain to which it should apply. I find more compelling research whose implications are road tested or at least employs a methodology which has a natural closeness to normal learning contexts.

5 In the U.K., we seem to have given up on the idea of teaching children to learn about

probability below the age of 11. The movement in *modern maths* seemed to establish probability as a valid topic for study in mathematics. Those developments led to its inclusion in the Primary school curriculum. However, in the recent versions of the National Curriculum for England and Wales, probability has almost been abandoned (and indeed the study of combinatorics). It seems reasonable to assume that this is as a result of disenchantment through lack of success in teaching this topic to children under the age of 11.

6 The use of the word *similar* is complex. A main thrust of situated cognition is that tasks presented in different settings are by definition not similar. The different structuring resources make the tasks essentially different. So, similar is used here to denote that the tasks contained the same mathematical structures as could be identified by an expert.

7 There have been many such microworlds. I have personally been involved in some of these developments (Pratt, 1992; Pratt & Ainley, 1989).

8 More recently, there has been a call to integrate the E.D.A and probability-centred approaches to the teaching of statistics on the basis that each in isolation offers a too narrow picture of statistical inference (Biehler, 1994).

9 A notable exception is one study which found evidence of the representativeness and availability heuristics, together with the equiprobability bias and the outcome approach, in children aged 11 and 12 years and discussed the influence of their culture on the use of these heuristics (Amir, 1994).

CHAPTER THREE

Aims of this Study

3.1. CENTRAL THEMES OF THE THESIS

There are two related overarching themes which run throughout this study:

- the articulation of certain principles and heuristics to describe and rationalise the design of a computer-based domain for stochastic abstraction, and
- the child's construction of meanings for the behaviour of stochastic phenomena.

The top-level intention of this thesis is to ask whether it is possible to devise an approach which introduces formal expressions of mathematical concepts into informal domains of learning. The observation of students' evolving thought in a carefully designed domain may provide a better understanding of how the specific features of the domain shape and are shaped by activities within it. Such an approach, if possible, suggests a dialectic in which the informalising of the formal and the formalising of the informal co-exist, blurring the notion of the formal and the informal as separate expressions of stochastic knowledge.

Chapter Two on the design of computational environments provides evidence that such an approach might be possible, since the computer offers access to formal ideas in a concrete way by representing abstract mathematical ideas in iconic form on the screen, facilitating direct manipulation by the user.

I will construct a setting in which individuals will meet the consequences of their beliefs. My aim is to build a domain of abstraction in which the laws of probability matter, in which it is possible to *work with* these formalisations, rather than to approach the formal as a separate domain grafted on to activity. My intention, then, is to put individual learners in situations where they can express their beliefs in symbolic (programming) form. Through the articulation of the beliefs that they hold, it is hoped that learners will be able to reconstruct them in the light of their experiences.

The principles which underlie the construction of such a domain for stochastic abstraction are not self-evident, and moreover intimately structure and are structured by considerations of the epistemology of the stochastic and of the pedagogy involved in teaching and learning about randomness. The articulation of these design principles is a second broad aim of this study.

The two main themes of this study, the articulation of principles underlying the design of a domain for stochastic abstraction and the child's construction of meanings for stochastic phenomena, are seen as inter-dependent since the structures and tools which are built into the software will inevitably shape the learner's construction of meaning. At the same time, the tools and resources will themselves be an articulation of principles drawn from the observation of children's expressions of stochastic intuitions.

In order to explore these two themes in more detail, and in particular to articulate the specific aims pertaining to these themes, each in turn will be discussed but it should be understood that the *global* aim of the study lies somewhere in their interaction. In the next section, I consider those aims which relate principally to the first of the these two themes, which I express as the construction of meanings *for* a stochastic domain. These aims concentrate on the particular features of the domain and how they influence the construction of meaning. The subsequent section considers those aims which relate to the second theme, which I refer to as the construction of meanings *in* a stochastic domain.

3.1.1. The Construction of Meanings For a Stochastic Domain

The study will explore the principles and design heuristics used in the process of constructing a computer-based microworld. diSessa (1985), in the early stages of designing and developing a new computational medium[10], made the case for examining this domain more closely:

> This paper derives from the conviction that trying to articulate principles in the process of design can advance the state of the art by developing explicit and testable general ideas in a context where the actual impact of those ideas in selecting and generating design alternatives is visible.
>
> (p. 3)

The approach draws its inspiration from the notion that it should be easier to analyse and make sense of the design process as it is acted out rather than through an examination of the final product. An iterative design process will allow the study of the gradual development of the computer-based microworld as well as the way that the subjects use the emerging software. This approach is discussed in more detail below but this first glimpse enables me to articulate the following aim:

<u>I wish to articulate the principles underpinning the design of a domain for stochastic abstraction and the extent to which these principles embody epistemological and</u>

pedagogical considerations.

This statement of intent can be expressed more explicitly through the asking of three questions:

(i) What do formalisms of the stochastic look like in a domain which emphasises meaningfulness? In particular, does the priority of meaningfulness have costs in terms of rigour, generality or economy?

(ii) What structures should we embed in the domain for stochastic abstraction in order to facilitate the articulation of intuitions about the stochastic?

(iii) What structures will facilitate the forging of new connections between intuitions and formalisms?

3.1.2. The Construction of Meanings In a Stochastic Domain

The study will also explore how young children's notions of chance and probability develop as they build their own products within a computer-based setting. The computer setting will enable the detailed study of children's intuitions through the articulation of those intuitions during activity within the microworld using a conventional programming language. I am not, of course, able to observe the learner's thinking directly but I am able to study and analyse the learner's actions as they are played out within an expressive medium on the computer. In this sense, the computer acts as a window for the researcher, on the children's intuitions of the stochastic. At the same time, for the learner, the computer can be regarded as a window through which the mathematics built into the system[11] can be observed.

Other authors have written about the metaphor of computer as window (Noss & Hoyles, 1996) In fact, this notion of *window* is not an attribute of the computer per se but rather depends critically upon the nature of the software being used (and more broadly on the activity which defines the use to which the computer and the software are directed). In the case of the software, the higher the degree of expressiveness[12], the more transparent the window. If the medium encourages the learner to express himself, the researcher may gain more detailed and meaningful data, whilst the learner, in attempting to make ideas explicit through the expressive medium, may in fact make new and increasingly meaningful connections.

In this sense, the study will examine the webbing of stochastic intuitions with computer-based tools i.e. I will explore how the children make choices about which structures are useful to them and how they use those structures to construct mathematical meanings in stochastic situations.

In the light of the research literature, one might conjecture that young children will already possess certain heuristics, primitive intuitions that they use to make sense of stochastic situations. One aim of this study will be to explore how the children's own intuitions merge with tools and structures into the software to constitute a web of internal and external resources.

I am now in a position to state explicitly a second broad aim which reflects this strand of the study.

<u>I wish to identify the nature of the intuitions that children use to construct meanings for random behaviour, and how these are shaped by the structuring resources in the environment. In particular, I wish to examine the extent to which the newly constructed meanings are contingent upon these resources.</u>

This statement of intent can be brought into sharper focus in the form of three questions, which the study sets out to answer:

(i) What are the informal intuitions of stochastic behaviour which children use and how are they cued by the specificities of the setting in which they act?

(ii) What situated abstractions are forged through the webbing of these intuitions and the structuring resources embedded in the domain for stochastic abstraction?

(iii) What are the features of the webbing process which determine the extent to which these situated abstractions become tools for the forging of new connections in related activity?

The next section of this chapter examines in broad terms how these aims will be achieved, whilst the next chapter, on the methodology of the study, examines the approach in greater detail.

3.2. AN OUTLINE OF MY APPROACH

In this section, I sketch with broad brush strokes the approach that I will adopt in order to study the above aims. Chapter Two discussed the deep relationship between children's construction of meanings in any mathematical domain and the tools which are available to mediate that conceptual change. This research sets out to study this relationship by focusing on *contingent change*.

3.2.1. Contingent Change

Contingent change then is a term which I apply to both strands of the study. On the one hand, the tools, and more broadly the domain which incorporates those tools

[13], are designed to bring about perturbations in the children's thinking, enabling the observation of their attempts to make sense of a shifting world. On the other hand, by observing children making use of such tools, and appropriating those tools by modifying them according to their own meanings and purposes, there will be changes in my own thinking which will in turn be encapsulated in the next iteration of the software.

Noss and Hoyles use the term *thinking-in-change*: (Noss & Hoyles, 1996) :

> ... we can set thinking in motion, and try to study what happens; we can set ideas in turbulence and investigate how changes occur; we can introduce new notions and try to understand how the thinker connects these with what he or she already knows.
>
> (p. 9)

I borrow the notion of thinking-in-change and, in using instead the term 'contingent change', I apply it more broadly in two main senses. I wish to place some emphasis upon a range of factors in the environment which may shape the construction of meanings for stochastic phenomena. In my study, I wish to examine the co-construction that takes place between pairs of children and myself. When the focus is on one child, then the actions of the other child and my own interventions can be seen as contingencies which perturb the child's thinking in ways which may in some respects parallel the influence of the computer-based tools. On the other hand, when the focus is on myself, the actions of the children can be thought of as contingencies which shape my construction of meanings for the domain of abstraction. The notion of contingent change is intended to emphasise the interrelatedness of the two themes of this study and the role of outside agencies.

3.2.2. Windows on Meaning-Making

My intention is that the development of tools for children to use and modify will make explicit my own meanings for the children's activity; I hope that my aspirations will be embedded in the code that makes those tools operate. In this study, there is an important sense in which both the children's and my own thinking are contingent upon the tools developed and how they are used. In the next chapter, I will set out exactly how these changes will be captured and analysed, but the approach will be to use the computer as a window (Noss & Hoyles, 1996) on the construction of meanings by both the children and myself:

> By offering a screen on which we and our students can express our aspirations and ideas, the computer can help to make explicit that which is implicit, it can draw attention to that which is often left unnoticed...... the computer, as we shall see, not only affords us a particularly sharp picture of mathematical meaning-making; it can shape and remould the mathematical knowledge and activity on view.
>
> (p. 5)

The aim is that the children's actions as they select and make use of those tools, articulated through button presses, choices from menus, changes to the code, and discussions, will lend greater transparency to the meanings for stochastic processes even as they are being constructed by the children. I intend to provide children with quasi-concrete instantiations of devices with which they are already familiar for their everyday encounters with the stochastic. Let me provide a first glimpse at one such tool to assist the reader's appreciation of what I mean by a computer-based tool and how the use of such tools can act as a window.

Consider the simple example of a dice. We can assume that young children already have used dice in games at home and will have experience of how a dice behaves in such contexts. I can provide a simulation of a dice on the computer. The dice can be represented iconically so that it appears much like a real dice. The critical difference between the computer's dice and its everyday counterpart though is that the computer version can be controlled. The children can modify how the dice operates. The process by which children program the dice towards some further purpose can be traced through their use of the programming language, the choices that they make, and the discussions that are provoked. This is the stuff of the window which should enable me to gaze upon the construction of meanings in the stochastic domain. My interpretations of those actions, and the insights I gain, will then need to be expressed in the form of further developments to the dice and the interface which enables the child to use and modify the dice. There is a real sense in which the change from one version of the dice to another captures my contingent thinking on the meaning *for* a stochastic domain, a window on the process of articulating the principles which underlie the construction of that domain.

3.2.3. Iterative Design

By modifying the available tools of mediation, it should be possible to ascertain how those tools shape children's construction of meanings for stochastic processes. At the same time, the insights gained from observing children using and modifying those tools will feed into the next iteration of tool development. In a real sense, that next iteration of tools will encapsulate my emerging understanding of the relationship between the tools and the children's meaning-making. This approach has been called iterative design (diSessa, 1986) , a method of constructing a sense of the variation in children's thinking in a particular domain (diSessa, 1989):

> One must carefully observe and document the activities of children in prototypes of the proposed microworlds .... In the best circumstances, one can experiment with a broad range of activities and variations of the microworld, collecting phenomenology of child learning, including some sense of the span of conceptual states that children might be in.
>
> (p. 216-217)

Figure 3.1 sketches the iterative process in this study which aims to observe the dialectic relationship between two main themes.

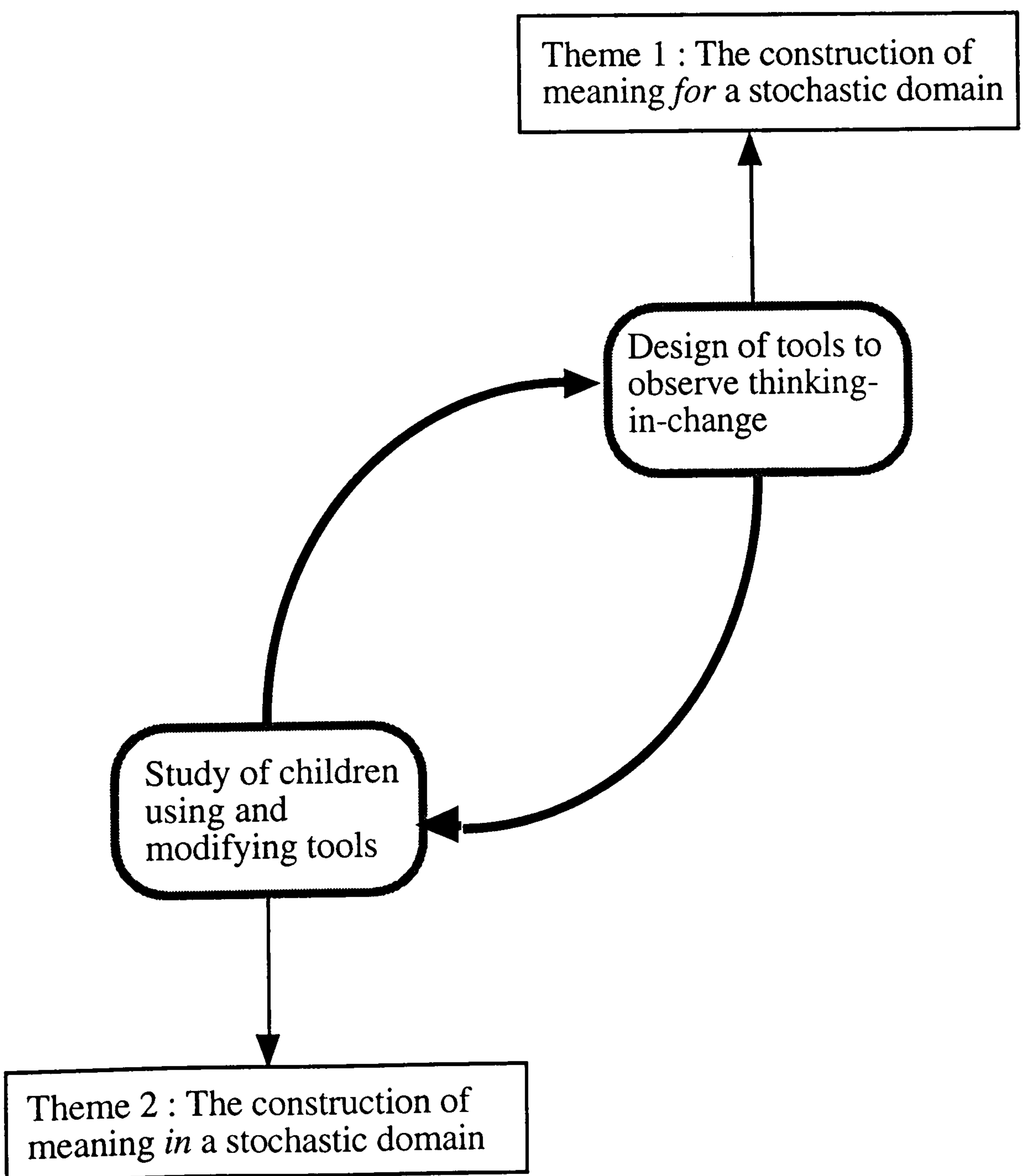

Fig. 3.1 : A sketch of the iterative design of the study

The focus will be on the intensive analysis of the interactions of a number of similarly aged children who will work for quite lengthy periods with computer-based tools. Though this approach will not enable the generation of theory relating to the construction of meanings for stochastic processes across a wide range of developmental stages, an aim which is beyond the scope of this study, it should provide a window onto the early stages of restructuring of children's naive intuitions about the stochastic. The iterative process will facilitate, through the gradual refinement of computer-based tools, an increasingly fine focusing on the primary issues involved in this restructuring process, including an enhanced appreciation of the initial state of those intuitions.

To help the reader make more sense of the role of iterative design in this study, consider once more the simple example of a dice tool. Suppose, at an early iteration, a fairly crude version of a dice tool nevertheless reveals an intuition that the dice should generate equal, or nearly equal, frequencies for its six different outcomes, even when the number of throws of the dice is quite small. It is easy to imagine how this might provoke some reflection on what sorts of structures and tools might be built into the next iteration so that the child can work on this intuition, and perhaps construct new meanings for the behaviour of the computer's dice over relatively small numbers of trials. Further iterations are likely to be needed in order to refine yet more the effectiveness of the tools for shaping such intuitions of the behaviour of the dice. In so doing, it is reasonable to suppose that insights will be gained about which features of those tools make them more or less effective in the webbing process.

3.2.4. Boxer — A Computational Medium

In pursuing these aims, the selection of a computer language for the development and use of the computer-based tools is a crucial decision, since the structures within the language will have a fundamental influence on what can be made available in the way of tools for children to use and on the way that those tools can be modified during the iterative design. After some deliberation upon various environments[14], I chose to work with Boxer, which diSessa refers to as a computational medium.

At this point, I wish to elaborate briefly upon three important characteristics of Boxer, as identified by diSessa (1995).

Nontrivial tool; nontrivial skills

The complement of this phrase, easy in/ easy out, suggests that an activity which makes few demands on the learner is likely to make little impact on that child's thinking. In diSessa's terms, tools which have wide applicability and demand some attention are likely to lead to deeper learning. My aim of studying conceptual change rather depends on there being change to study. If it were a trivial task to generate such change in the stochastic domain, we would expect there to be much more evidence of adults sophisticated in their understanding and use of chance and far less evidence of the misuse of heuristics for making judgements of chance (see Chapter Two). We must therefore assume that the environment will need to demand a considerable degree of engagement on the part of the learner, which explains the relevance of the nontrivial tool/ nontrivial skills feature.

Exponential cumulativity

Skills which have been learned should be applicable to a wide range of situations. In this way, the skills become cumulative. diSessa refers to the notion of *tuning*. Thus, a highly-tuned tool will fit a specific activity very closely but it will have little relevance outside of that activity. In contrast, a tool which is de-tuned will have much wider application though it may be more difficult to learn in the first place. However, the principle here is that non-trivial de-tuned tools may lead to the development of non-trivial skills. Although the scope of the study will not extend to the longer term, which diSessa has in mind here, there is still some relevance in his point. The children will be working for extended periods with these tools and the applicability of these tools to a range of stochastic situations will not only reduce the new learning that has to take place in each new situation but may also optimise the synthesis of meaning across different situations.

"Small dollops", ownership and deep appropriation

By designing highly-tuned tools, the user is required to learn each tool afresh; each will have its own syntax and narrow domain of applicability. These "small dollops" will be quickly used and thrown away; they are unlikely to be appropriated by the user. In contrast, diSessa argues that, by providing de-tuned tools, it is possible for the user himself to tune the tools to their specific needs, thus encouraging a sense of ownership. At the same time, the de-tuned tools can then be relatively easily re-tuned for a new situation. However, if the tools are de-tuned in the extreme, they may prove to be over-generic, and so lack understandability. It may be difficult for the learner to make sense of such tools. One aim of this theme within the study will be to observe the development of such tools so that by observing how they appear to shape the activity, it may be possible to develop this heuristic further, perhaps gaining a more sophisticated understanding of the balance between tuned and de-tuned tools. The notion that de-tuned tools can be tuned in use is extended in this study to the provision of tools. Boxer contains a re-constructible interface which facilitates the re-design of any Boxer-based environment. The ease with which tools can be modified will encourage experimentation through successive iterations.

3.3. CONCLUSIONS

I wish to conclude this chapter by re-stating even more specifically the aims of this study. I set out below two sets of questions; the corresponding aim in each case is to seek an answer to the question. The first set of questions refers to the construction of meanings *for* a domain of stochastic abstraction.

A1. In designing a Boxer-based domain which emphasises meaningfulness, rather than rigour, generality or economy, what do formalisms of the stochastic phenomena, such as randomness and distribution, look like?

A2. When children use Boxer-based versions of artefacts like dice and coins, what structures should we embed in the domain for stochastic abstraction in order to facilitate the articulation of intuitions about how those artefacts behave?

A3. Through the iterative design, which structures will facilitate the forging of new connections between intuitions (e.g. of randomness) and formalisms (such as representations of distribution)?

The second set of questions refers to the construction of meanings in a domain of stochastic abstraction.

B1. When young children interact with Boxer-based representations of everyday stochastic phenomena (dice, coins, spinners .....), what expressions of their informal intuitions of stochastic behaviour do we observe?

B2. How do the structures within the domain of stochastic abstraction, built into Boxer, support the forging of situated abstractions?

B3. What are the features of the webbing process (including the children's intuitions as articulated within the Boxer microworld, the tools provided and the influence of other agencies such as other children and my own interventions), which determine the extent to which these situated abstractions become tools for the forging of new connections in related activity within other parts of the Boxer-based domain?

In the next chapter, I discuss in much further detail (than was appropriate in the above broad sketch of the approach) the methodology which will enable me to study systematically these aims.

NOTES

10 diSessa heads a team at the University of California, Berkeley, which is developing what is termed a computational medium, Boxer. This project is seen as extending the notion of literacy to a new domain, where users express themselves in various ways, including mathematically, in various modalities (graphic, literal, computational, ...) within the Boxer medium.

11 From a constructivist perspective, the notion that mathematics can exist independently within the software is invalid. In fact, I use the phrase here metaphorically to mean that the programmer, in designing the microworld could envisage certain plausible actions upon the structures that he intends to offer. In carrying out this virtual experiment, the microworld designer selects those structures where the plausible actions appear to lead typically to the construction of mathematical meanings. This fuller translation of the phrase makes the case for why the metaphor is useful and why it will be used throughout the study.

12 A medium is expressive to the extent that it includes structures which can be used to combine, sequence and manipulate objects in order to create new objects, which themselves might then become the manipulated objects. Such objects may be iconic, such as the points and shapes in a dynamic geometry package, or they may be symbolic, such as a word like FORWARD in Logo. Expressiveness is also a function of understandability. diSessa (1985), for example, in discussing the design principles for Boxer, places much emphasis on his notion of naive realism, by which he means that the user is able to believe that he or she is actually directly manipulating objects, even though in reality they are in fact merely transforming on-screen representations of those objects.

13 This domain includes a number of levels beyond the computer-based tools. At one level are the tools which I have specifically built into the computational medium. At another almost indistinguishable level are those tools and structures already available in this medium to both myself as tool-builder and to the children as tool-users and modifiers. In addition, an individual child will find support from the interactions with a co-worker and from the interventions thrown into the sense-making process by myself. Another significant aspect of the domain is the task to which those tools are directed.

14 Other contenders included various versions of Logo such as LogoWriter, Microworlds Builder and StarLogo. The latter was particularly attractive because of the special features related to the use of multiple parallel events, as exemplified beautifully in Wilensky's PhD thesis (1993).

In the end the decision favoured Boxer largely because of the flexibility it offered in interface design.

CHAPTER FOUR

Methodology in an Iterative Design

4.1. OVERVIEW

This chapter is divided into five main sections: iterative design, the setting, a qualitative methodology, data collection and data analysis. Below I provide a brief overview of each of these sections.

- *Iterative design*

  The rationale for an iterative approach has been discussed in Chapter Three. This section sets out a post hoc stratification of the study into four iterations, each characterised by a phase of software or tool development, followed by a phase in which children used and modified those tools. There is also some top-level discussion about the relationship between these iterations.

- *The setting*

  This section briefly describes the environment in which the tool-use phase of the research took place.

- *A qualitative methodology*

  This section gives a brief overview of the underlying reasons for adopting a qualitative approach and in particular discusses the role of the participant observer.

- *Data collection and associated methodological issues*

  The methods of data collection are described. A number of important methodological issues relating to these methods of data capture are also discussed.

- *Data analysis and associated methodological issues*

  The methods used in each iteration to analyse and generally process the data are described. Methodological issues underpinning the use of case studies is discussed.

Each iteration has two phases, tool development and tool use, roughly equivalent to the two themes in the previous chapter; I describe these two phases below.

4.1.1. Tool Development

The tool development phase of an iteration is marked by the modification of previously designed Boxer-based tools and the development of new tools. Apart from Iteration 0, tool development emerges out of the analysis of children's use of the tools made available in earlier iterations, especially that immediately preceding the current iteration.

4.1.2. Tool Use

The tool use phase focuses on the use and modifications of tools by children. The choices made, the ways in which the tools were used, and the relationship between the tools and the expression of intuitions about the stochastic, including the discussions that such tool usage generated, are examples of data generated by the tool use phase of each iteration.

4.2. ITERATIVE DESIGN

The study developed through four iterations: A first (post hoc) glimpse of each iteration is given in Table 4.1. These descriptions will serve to explain how the methodology itself developed in the light of outcomes from previous iterations.

Iteration 0	**Iteration 1**	**Iteration 2**	**Iteration 3**
Bootstrapping	*Exploratory*	*Developmental*	*Analytical*
This iteration began with no prior tools except those already available within Boxer. The aim was to initiate the process by making a first guess at what appropriate tools might look like and trying them out on a pair of children.	The bootstrapping iteration gave rise to some early ideas of how those tools might be developed and some additional tools that might be needed. The emphasis is more on the development of tools than on their use.	This is a stage in which the tools are sufficiently advanced for there to be meaningful progress in our understanding of how these tools shape children's intuitions of chance.	The final iteration involves only relatively minor modifications to the tools but much more careful and systematic analysis of how the tools have impact upon the children's intuitions.

Table 4.1 : A rough description of the four iterations

These rough descriptions of the four iterations give an image of a switch in attention from tool development to tool use. (In Figure 4.1, the density of the shading represents the degree of attention to the use of tools in relation to their development.) In the beginning, the tools were inevitably crude, their design being based on an initial interpretation of the literature together with the evocation of memories of personal experiences with children, using computer-based tools in other mathematical domains[15], or non-computer-based tools in the stochastic domain[16].

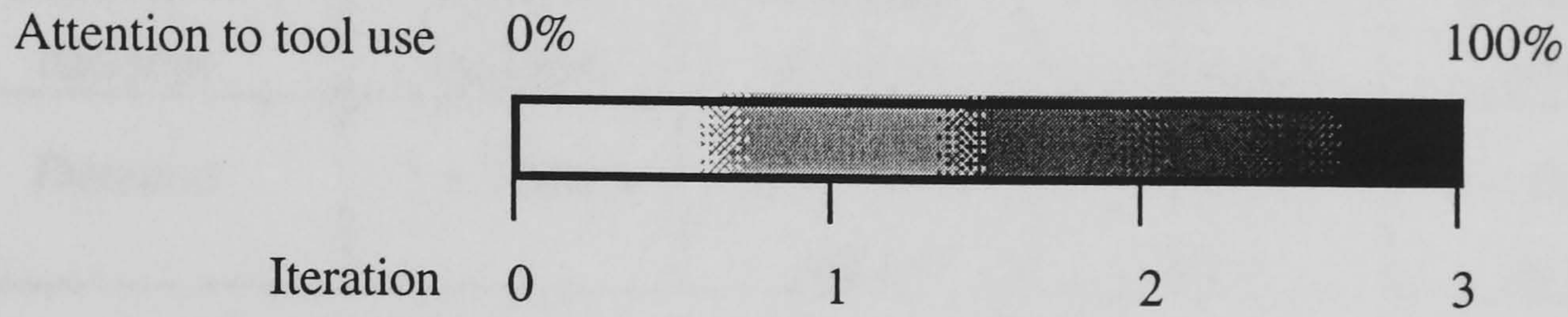

Fig. 4.1 : The changing emphasis from tool development to tool use

Predictably, early tool development was highly sensitive to the experiences of tool usage in the sense that relatively small amounts of such data generated extensive fresh tool development. In contrast, as the tools became more sophisticated and tuned, quantities of data from children's work prompted only minor changes to the computer-based tools. Indeed, the lack of change in the software was a useful heuristic signalling that the iterative process was converging towards a fairly stable design.

The tool-use phase was conducted through a clinical interview preceded (except in the case of Iteration 0) by a semi-structured interview. The methodological details are discussed later but the overall schedule is set out in the Table 4.2.

	Iteration 0	**Iteration 1**	**Iteration 2**	**Iteration 3**
Tool Development Phase				
Duration	April 1995 to June 1995	October 1995 to Nov. 1995	June 1996 to January 1997	June 1997 to July 1997
Tool Use Phase				
Date	June 1995	Dec. 1995	January 1997	July 1997
Semi-Structured Interviews				
Number of children	0 children	2 pairs of children	8 individual children	16 individual children

Duration	-	2 x (20 to 30) minutes	8 x (20 to 30) minutes	16 x (20 to 30) minutes
Clinical Interviews Number of children	1 pair of children	2 pairs of children	4 pairs of children	8 pairs of children
Duration	1 x 2 hours	2 x (2 to 3) hours	4 x (2 to 3) hours	8 x (2 to 3) hours

Table 4.2 : The schedule for the iterative design

There is an exponential increase in the number of children involved at each iteration, reflecting the move from exploration to more systematic analysis. The interviews held in the early iterations aimed to gain a general impression of what sorts of intuitions of the stochastic children may have at this age. Later the interviews tried to separate out the pre-existing intuitions of individual children and so it was felt necessary at that point to interview children on their own.

Initially, there was some doubt whether it would be better to hold the tool use sessions with individual children or in pairs. The first option would have the benefit of individualising the data. The alternative would gain in that the children might discuss issues as they arose with one another in a less constrained manner than might have been the case if each child were treated individually. In fact the early iterations with children working in pairs was so successful in terms of capturing spontaneous discussion that this format was kept for the remainder of the iterations. The individual interviews helped to some extent in separating or comparing the children's developing intuitions at the analysis stage. One aim of my questioning during clinical interviews was to separate the intuitions of the individual children.

4.3. THE RESEARCH SETTING

4.3.1. Age of Children

All the children were aged between 9.11 years and 11.1 years of age. The choice of this age range was based partly on pragmatic grounds and partly on theoretical considerations, as explained below:

- The setting, described below in more detail, enabled the use of Boxer by technologically-sophisticated children. The primary school involved has on its roll children up to the age of 11[17]. By deciding to work with children from this school, the distraction of technical obstacles was minimised allowing a tight focus on the children's intuitions of the stochastic.

- Another important advantage of working with children from a primary school was curriculum-based. The National Curriculum for England and Wales contains almost no work on chance and probability at this level. It is not until secondary school that children begin to study these areas with any degree of depth. The lack of any previous systematic study benefited the research in so far as pre-existing intuitions were likely to be based on the informal abstraction of everyday experiences.
- A related aspect of working with these children was one of pedagogy. The teaching and learning approach used in this school is described in the next section, but the impact on this study was that, not only was there little evidence of formal teaching, but also the children were likely to be well versed in problem-solving process-oriented approaches to learning, skills which they used in the tool-use phase of each iteration, and which enabled the observation of the learning process.
- At the same time, there was reason to predict that this age was particularly ripe for conceptual change in this domain. It was reasonable to suppose that the older the children the richer was their stock of everyday experiences of the stochastic. These experiences may have perhaps led them towards the abstraction of many heuristics for making judgements of chance, which were in effect the raw material to be studied in change through the window of the computer screen.

4.3.2. Technical Expertise of the Children

The data in this study was collected alongside the ongoing research of the Primary Laptop Project, in which researchers from the Mathematics Education Research Centre at the University of Warwick were studying the effects on young children's mathematical learning when they had continuous and immediate access to portable computers. Although this research was a separate study, it in effect piggy-backed on the work of the Primary Laptop Project, which had created the environment which made this study possible. The children, whom I have studied, were part of the Primary Laptop Project and so it is worth giving some background information regarding its methodology which inevitably influenced this study.

In the Primary Laptop Project, the computers were seen as part of a complex working environment, where many aspects came together to support the children's learning. Prior to and throughout this study on intuitions of the stochastic, several classes of children, aged between 8 and 12 years, had been using portable Macintosh systems throughout the curriculum and for as much as two thirds of the year. The children in this study were drawn from these classes. The machines were generally shared between two children. Ownership of the machines by the children, and parental involvement, were encouraged by a number of strategies.

- The children were expected to look after their machine e.g. they had to make sure it was put away correctly, re-charge the battery, decide who took their machine home, etc.
- The children often chose when to use the machine in school. Decisions not to use the machine were respected just as much as their choice to use it whenever they wanted. The exception to this rule was that the teachers and researchers often designed activities which required the use of the computer.
- As far as possible, the children were expected to decide how to maintain the desktop and their own folders for saving their work.
- The software on the machines, including the more gimmicky aspects (our description — not necessarily the children's), was there to be used whenever seemed appropriate. We avoided systems which over-protected the child in the name of protecting the software.
- The children were encouraged to demonstrate their work to their parents when they took their machines home. Indeed, the children seemed to gain much from this process, especially as they often ended up tutoring their parents.
- The children were often put into the role of tutors. For example, from time to time, the children in the project handed their machines over to a new class. If the new class were not already familiar with using the hardware or the software, the experienced children would tutor them into this way of working. This peer tutoring became extended to specific activities where one class would show another class their projects which had emerged as a result of their work on the project.

In the current study of intuitions about the stochastic, the children were very familiar therefore with using the graphics, spreadsheet and word-processing components of *ClarisWorks* and with *LogoWriter* both during lessons and in their private use of the laptops at home. They were not however familiar with *Boxer*, though the experiences with LogoWriter had introduced them to the notion of a computational medium. Some children in this study had a brief introduction to Boxer before working with the tools provided for this study. Others came completely new to Boxer. In fact even these children had no significant technical difficulties with the tools, which must, in large part, be attributable to the wealth of technological experience that these children had already accumulated.

4.3.3. Pedagogical Stance of the Class Teacher

The normal class teacher for the children in my study was herself a participant in the Primary Laptop Project. In the use of the portable computers with the children, she adopted a broadly *constructionist* framework (Harel & Papert, 1991). In her contribution to the project, she would co-operate with other teachers and the researchers involved in the project to plan activities within which were embedded mathematically powerful ideas. The children were encouraged to work on projects, developing an independence from the teacher but at the same time sharing their work and their ideas with each other. Such sharing, often guided by the teacher, helped to stimulate reflection on the important mathematical themes arising from the work.

In this stage of the laptop project, the research was essentially exploratory, rather than addressing clearly focused research questions. The main interest lay in exploring the range of mathematical activities that were possible for children in this environment, and in identifying areas for more focused research in the future.

This pedagogy was consistent with the research methodology used in the research for this study. I negotiated with the teacher appropriate times to take the children out of their normal class into an adjacent room. The mutual trust between her and myself, developed during the Primary Laptop Project, made this a relatively straight forward process. She recognised that the distinction between research and teaching within this methodology was not nearly as stark as it might have been under other more conventional methodologies. For example, the research approach was not inconsistent with a pedagogy which stresses the importance of children constructing their own artefacts and ideas. Similarly the interventions (discussed in more detail below) designed to probe into the children's thinking, and indeed occasionally to perturb that thinking, are not inconsistent with a teacher adopting a constructionist paradigm. I am not wishing to suggest that there is no distinction between the role of the researcher and the teacher, indeed there are substantial differences, in aims for example, but I merely suggest that the pedagogy of the teacher and the methodology of this study had some similar features which served to support the research process.

4.4. A QUALITATIVE METHODOLOGY

The methodology was determined to manage three complex issues.

(i) The study set out to explore two interrelated strands side-by -side, the construction of meanings for and in a domain of stochastic abstraction.

(ii) While there has been an explicit research effort in mathematics education to advance our knowledge in one of those strands, namely theme 2, the principles and heuristics of software design is a relatively new knowledge domain[18]. Methodologies for such endeavours have not yet settled down into conventional and accepted forms.

(iii) In the other theme, children's construction of meanings for stochastic processes, the focus of attention has primarily been on the cataloguing of misconceptions. These studies offer little guidance as to how one might systematically research conceptual change in this domain.

In view of this complexity, I decided to use a qualitative approach to research the aims of the study. The style of the research was planned as an exploration of the evolving construction of meanings for stochastic processes as might be inferred from the children's actions within an environment, which was itself changing in response to those observations. This study did not set out to test predetermined hypotheses. The rationale for an iterative approach has already been discussed in Chapter Three but I can now set this approach within a broader methodological paradigm.

4.4.1. The Role of Participant Observer

It was determined that a process-oriented qualitative approach would be appropriate to identify types of intuitions for stochastic processes and how these change during extended activity. The focus has been on description and interpretation, in the ethnographic or anthropological tradition (Goetz & LeCompte, 1984) rather than on measurement and prediction.

My own role was seen as that of a participant observer, interacting with the children in order to probe the reasons behind their actions, later interpreting these reasons in the light of observations based on other children's work. The researcher's aim, when acting as a participant observer, has been described as follows (Burgess, 1984):

> it is the researcher's aim to compare these accounts (i.e. those obtained from informants) with each other, and with other observations that the researcher has made in the field of study.
>
> (p. 79)

In the role of participant observer, there were times when it seemed appropriate to make teaching-type interventions to perturb further the children's thinking (see the section on *contingent thinking* in the previous chapter and the section on the role of my interventions later in this chapter). It has been argued that (Burgess, 1984) :

> Participant observers are involved in face-to-face relationships with those who are researched, and that the observers are part of the context that is being observed. This results in the possibility of researchers modifying and influencing the research context as well as being influenced by it themselves.
>
> (p. 79)

Such interventions were carefully documented, being recognised as another type of data for subsequent analysis, especially in pursuing the aims relating to the construction of meanings in a stochastic domain.

4.5. DATA COLLECTION AND ASSOCIATED METHODOLOGICAL ISSUES

4.5.1. The Tool Development Phase of Data Collection

This section considers the methods used in order to collect data relating to the development of computer-based tools and structures. Data for the analysis of this phase of each iteration took three forms:

Versions of the microworld

The software developed through versions, which can be identified, post hoc, by the emergence of a new way of thinking about the nature of the tools or the relationship between them. These versions roughly matched the iterations but not entirely so. The rough match was predictable in the sense that it was the work with children that often provided fresh insights and so a phase of new development in the tools. Sometimes however the insights were delayed, bringing about a break-through in the development part way through the next iteration so that there was not a perfect correlation between iterations and versions of the software.

The information in Table 4.3 gives an impression of how the development of tools relates to the iterations. The significant break-throughs are indexed by changes in the integer part of the version number.

Iteration Number	Version Numbers
Iteration 0	Version 0.01 to 0.26
Iteration 1	Version 1.01 to 3.21
Iteration 2	Version 3.22 to 4.20
Iteration 3	Version 5.01 to 5.30

Table 4.3 : The relationship between iterations and versions

Diary entries

I maintained a diary, recording minor developments, significant breakthroughs and trying to capture the issues that gave rise to that progression.

Communications

There was regular communication, sometimes face-to-face, often through eMail, between myself as the programmer and Professor Richard Noss, as co-developer. The process of trying to communicate half-understood ideas in written form often resulted in the clarification of those ideas and generated a source of data to look back upon at a later date.

4.5.2. The Tool Use Phase of Data Collection

This phase of each iteration included two different data-collecting techniques:

Semi-structured interviews

During each iteration the children were interviewed prior to their work on the computer (except in the case of Iteration 0). These interviews were audio recorded and later transcribed. They aimed to illuminate the children's intuitions for the stochastic at the start of the activity. A number of catalytic questions were put together but these were designed to prompt rather than limit discussion. The nature of the questions changed as the issues became increasingly apparent through the iterations but the style adopted was invariably one which was prepared to respond in the moment to the children's replies so that the framework was not seen as rigid but as flexible, increasing opportunities to seek out more precisely what it was that the children were thinking. The schedules for the semi-structured interviews in Iterations 1 to 3 are included in the Appendices.

Clinical interviews

The tool-use phase of each iteration was explored through the use of a clinical interview. The schedules for the clinical interviews in Iterations 1 to 3 are included in the Appendices.

In order to achieve faithful recording, the clinical interviews were video-taped (except in the case of Iteration 0 when the clinical interview was audio-taped). The actions played out on the screen were transmitted directly to videotape and the children's discussions and my interventions were overlaid onto the videotape as audio-recordings. Figure 4.2 shows the set up of the equipment.

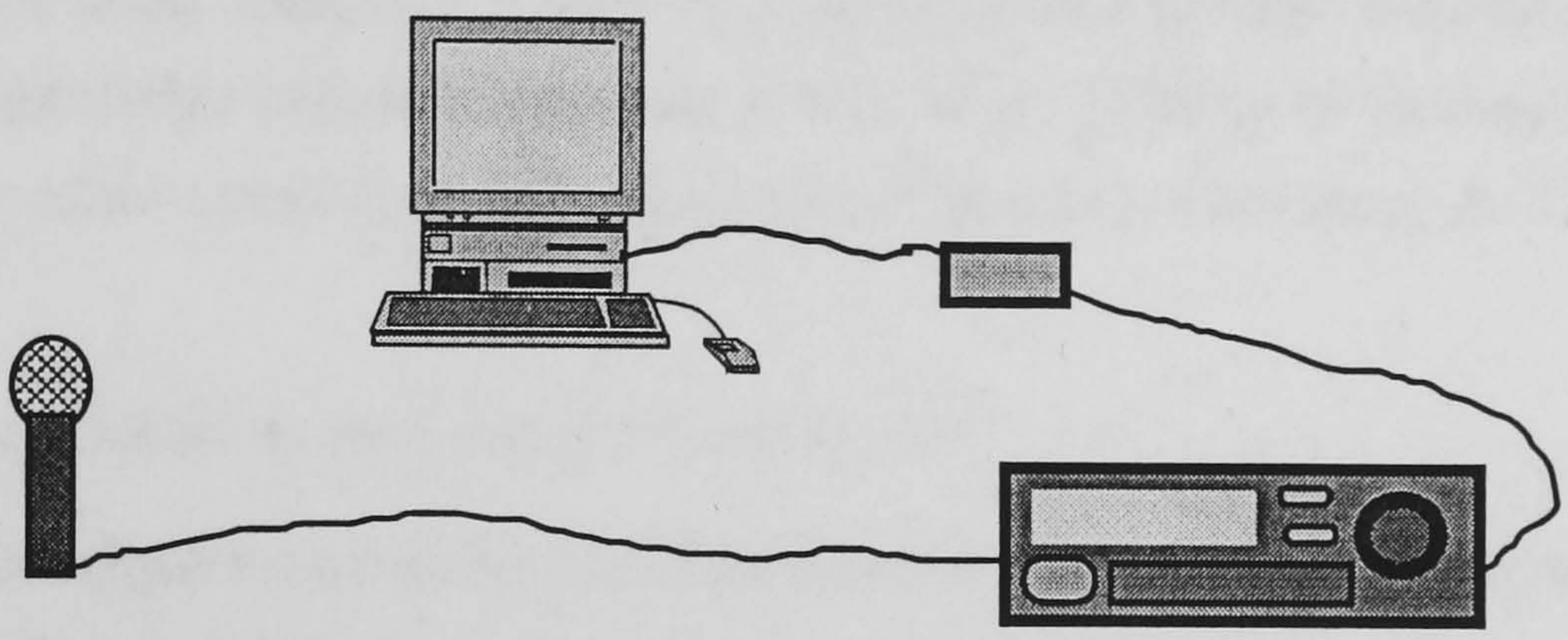

Fig. 4.2 : The recording of clinical interviews

This system enables the recording of every screen-based action and the consequent discussions but it did not record the faces or bodies of the children with some inevitable loss of data, relating to such issues as body language and movement of hands. When this appeared to be very important, brief field notes were kept. In fact, the children often used the mouse to point to objects on the screen, which helped the recording, and the nature of my questions often elicited responses, which clarified actions that may otherwise have been open to many interpretations.

The clinical interview has been described as follows (Hunting & Doig, 1995) :

> In a clinical interview, a dialogue or conversation is held between an adult interviewer and a subject. The dialogue centres around a problem or task which has been chosen to give the subject every opportunity to display behaviour from which mental mechanisms used in thinking about the task or solving the problem can be inferred.
>
> (p. 114)

Hunting suggests in this paper that clinical interviewing is not dissimilar to aspects of the teaching process. Certainly even the most innocent of interventions, aiming to identify the reasons behind a particular action, is likely to focus the child's thinking in a new direction, which is often the teacher's explicit aim. My methodology accepted that this must be the case and recognised that activity with computer-based tools was likely to increase opportunities for such teaching-type interventions, and saw the exploration of the effect of such interventions as part of the study (see below for further discussion of the role of my interventions).

The validity and reliability of such approaches has been fiercely contested. Certainly the notion that people's verbal reflections are simple translations of thinking, as if the mind were so transparent, has been long since discredited. In this study, the discussion related to mathematical activity, and there was reason to believe that, at least in this domain, the verbal reflections possessed some inherent validity. The children seemed to be rather good at telling what they believe and think about mathematical ideas. Indeed, it would be difficult to make sense of a claim that a child had knowledge independent of an accompanying ability to articulate it in a language or other symbol system (Swanson, Schwartz, Ginsburg, & Kossan, 1981).

This study exploited several forms of expression:

- Discussions between the children when my presence seemed to have been largely forgotten;
- Discussions between myself and the children, which were often used to validate and probe more deeply into the thinking behind their actions and discussions;
- The button clicks, menu choices and various ways of pointing on screen;
- The use of a Logo-like language, a formal symbolic expression of mathematical thinking.

These different forms of expression gave a rich picture, not only of the performance, but also of the thinking that stimulated or arose out of those actions. They often combined to give more confidence in the interpretations, a form of internal triangulation.

Swanson also claims that it is reasonable to rely on subjects' descriptions when the grain-size of the data is not too small (e.g. descriptions of problem solving in contrast say to the neural processing). Since my focus was on meaning, it seemed very reasonable to rely on a variety of forms of expression.

We must also note that the act of reporting verbally may itself affect the mental processes. Swanson emphasises the negative aspects of this effect in the sense that he claims that reporting may get in the way of the mental processes. It was perhaps even more likely in this study that such reporting, since it generated reflection which may not otherwise have occurred, would draw attention to specific areas of mathematics and so actually improve performance. Either way, validity is only questioned if the experiment were seen as one in which the verbal reflections were somehow extraneous. In fact, in so far as the questions supported performance, they became an intrinsic part of the construction of meaning. The concern was not that the questioning caused changes but that they were recorded, and so available for analysis.

In fact there were subtle changes in the purpose of the clinical interviews as the iterations progressed. These changes can be categorised using Ginsburg's analysis (Ginsburg, 1981). He defined three purposes:

(i) The discovery of cognitive activities

Iteration 0 (bootstrapping) and Iteration 1 (exploratory) are very clearly situated in Ginsburg's first category. A process of discovery must be employed to determine the main developmental features of children's mathematical thought. The technique involved beginning with a task, asking contingent questions and requesting reflection.

(ii) The identification of cognitive activities

Discovery did not stop with the advent of Iterations 3 and 4. Nevertheless, there was increasing emphasis in these iterations of attempts to provide descriptions of the construction of meaning, what Ginsburg more generally calls "descriptions of the mind". The clinical procedure was now more focused and prepared to channel children's activity into particular areas. A greater degree of standardisation between different pairs of children emerged without compromising the flexibility of response characteristic of all clinical interviewing. It was possible in these iterations to make more use of what might be called experimental method.

(iii) The evaluation of levels of competence.

In Iteration 4, Ginsburg's third purpose was also evident. Some of the questioning sought to identify the maximal, as opposed to typical, level of competence that these children could achieve.

In the next section, I will discuss in more detail the role of my interventions, which were in fact influenced by these three purposes.

The role of my interventions

In general, the aim was to allow the children to be in control of their explorations, making decisions and moving in directions of their own choice. Nevertheless there were occasions when it seemed appropriate to make interventions. Such interventions were carefully recorded and became part of the data for analysis.

Interventions were offered, whenever possible, in a fairly open, certainly non-threatening way, still offering the children opportunities to interpret the suggestion in a variety of ways. On some occasions it was necessary to be more specific, such as when the initial intervention was not understood, or when the intervention was transparently technical in nature (see the first type of intervention below). Within this pattern of behaviour, my interventions fell into the following categories:

(i) Technical

Occasionally and less frequently as the clinical interview progressed, it was necessary to explain specific aspects of the software or hardware. This was most obviously the case at the beginning of the session when the interface had to be explained and, after some initial exploration, when I intervened to show how the gadgets could be opened up to reveal various additional tools.

(ii) Probing

The actions of the children were often less than transparent when it came to inferring the reasons or intuitions that might lie behind those actions. A fairly frequent type of intervention was one which sought to tease out the motivation for a particular action (see *discovery and identification of cognitive processes* in the section above).

(iii) Explaining

Occasionally the children turned to me for an explanation of some phenomena. I almost always turned such questions back to the children by asking what they thought, or on those occasions when the children asked, "what would happen if", by suggesting that they try it for themselves. Such an intervention was not entirely neutral since the children no doubt inferred that this was not a completely dead end direction to follow, and so were perhaps unintentionally encouraged to follow that route.

(iv) Experimental

These interventions were the closest to what we might think of as teaching. This type of intervention explicitly aimed to be non-neutral. It sought to make some change in the direction of the activity with possible implications for conceptual change. There were three main purposes for experimental interventions:

- Such interventions were sometimes needed because the children were simply going down what was clearly a dead end, which seemed to have no potential pay-off from either the research or learning points of view. Such interventions were dangerous in the sense that such predictions could not be fool-proof. What might have seemed to be a dead end may sometimes have had surprising benefits. There was nevertheless an acknowledgement in my methodology that there were times when a judgement needed to be made about allowing the children to continue in the direction on which they had set out.

  The methodology was not intended to be naturalistic as I did not intend to wait until something interesting might have happened. On the contrary, the approach was more experimental in the sense that, by choosing to use computers, I was hoping to initiate change in the child's thinking which I could then observe. As long as the motivation and the fact of the intervention was recorded and built into the analysis, I do not believe that there was any loss of validity.

- The second purpose for an experimental intervention was because the children were stuck. Sometimes this was obvious to the children and they sought advice as to how to proceed. At other times, the children were less aware that they were in effect covering the same ground, with little advantage to either them or the research, because this repetitiveness was disguised by apparent activity. On such occasions, I offered a suggestion which might lead to a new direction.

- A third reason for an experimental intervention, which was in fact more applicable to the later iterations, was to explore whether a child was able to work with a new idea. This type of experimental intervention was focused on competence (see *evaluation of levels of competence* in the section above). It addressed the question, "What is the maximal level of performance that the child's intuitions, supported by the computer's tools, can achieve?" Such interventions were not often relevant and were only used when a child seemed to be particularly confident and already performing with some fluency.

Ainley has discussed in more detail the role of the experimenter in relation to other work in the Primary Laptop Project (Ainley, 1995).

The discussion so far has only related to the process of data capture though these issues do have implications for the subsequent analysis. The next sub-section discusses methodological issues arising from the subsequent processing of the data.

4.6. DATA ANALYSIS AND ASSOCIATED METHODOLOGICAL ISSUES

4.6.1. The Tool Development Phase of Data Analysis

The development of new computer-based tools was forged out of six interacting processes:

Epistemological analysis

The mathematical knowledge domain was analysed to identify the main strands of knowledge that in their synthesis define stochastic processes.

Virtual experiments

I played mind games in which I imagined stereotypical children, who were presumably amalgams of all the children I had worked with in the past, using hypothesised tools. The outcome of such virtual experiments were roughly hewn sketches of what sorts of tools might address the strands of knowledge identified from the epistemological analysis.

Prototypes

These rough mental sketches of possible tools were too vague and too personal to be expressed. However, the process of programming in Boxer forced the sketches to become fine drawings, in the sense that the idea became reality in the form of Boxer code. The code never completely matched the sketch but emerged out of the sketch through the formalising process called programming.

Evaluation through discussion

Once generated, the prototypes were now externally available for discussion and criticism. Professor Richard Noss was the main contributor to this discussion but at various times other colleagues contributed, sometimes through formal seminars and at other times through informal discussions.

Evaluation though tool use

These prototypes were then debugged and tested through use with children, the second phase of each iteration, resulting in another cycle of the stages listed above. The evaluation through tool use also provided a validity check on the above stages of development since the aim was to construct tools which would be meaningful to children. The validity of the tools was tested by observing whether that was the case in practice. This 'proof of the pudding is in the eating' type of check is rare in mathematics education research and is, in my view, one of the strengths of iterative design.

It would be misleading to give the impression that each of these stages was uniformly significant through each iteration. Epistemological analysis was highly influential during the early iterations and gradually died off as the iterative process converged, though, as if to emphasise that even this image is too simplistic, it is worth pointing out, for example, that the pie chart tool, which proved to be immensely valuable, did not emerge until the final iteration[19].

4.6.2. The Tool Use Phase of Data Analysis

Table 4.4 summarises the methods used to analyse the raw data.

Iteration 0 The audio-tape of two children, Sandra[20] and Colin, was partially transcribed. Issues were identified which informed the next phase of tool development. The issues were so stark that subtle analysis of the transcripts was deemed unnecessary.
Iteration 1 The videotapes of each pair of children, Jasbir & Sunil, and Jenny and Gill, were partially transcribed. The transcriptions and associated field notes were used to develop tags that represented possible emerging issues and the transcripts were annotated using these tags. The main issues were developed and discussed with Professor Richard Noss by considering these tagged transcripts alongside the children's semi-structured interviews which had also been transcribed. A conference paper, (Pratt & Noss, 1996), made these issues more explicit for peer assessment.

Iteration 2 The videotapes of four pairs of children were partially transcribed. These pairs were: Adrian & David, Linda & Ruth, Shirley & Julie and Tim & Rupert. The transcriptions, field notes and semi-structured interview transcripts were used to develop tags as in the previous iteration. The main issues were developed and discussed with Professor Richard Noss. A number of seminars made these issues more publicly available with the result that further ideas were generated.
Iteration 3 The videotapes of eight pairs of children were partially transcribed. These pairs were: Terry & Joe, Ray & Luke, Donna & Rose, Steve & Richard, Lenny & Lee, Anne & Rebecca, Cathy & Lynn, and Neil & Gurdev. Case accounts were developed for each pair of children. The case accounts represented plainly told stories, avoiding as far as possible interpretations of the transcript. From these plain case accounts, I developed interpretative case analyses, in which various inferences were made as to why and how the children's construction of meanings developed. The case analyses drew on the transcripts of the pre-interviews as well as the case accounts of the clinical interviews. These case analyses were discussed with Professor Richard Noss to validate the inferences drawn. Issues were then identified by comparing and contrasting these eight interpretative case analyses. Similarities and marked differences were noted. All sixteen transcripts of the pre-interviews, all eight case accounts and the eight case analyses are available from the World Wide Web address: http://www.warwick.ac.uk/wie/staff/DP.htm

Table 4.4 : Data analysis in the tool use phase of each iteration

As indicated above, the videotapes from the clinical interviews were partially transcribed. These sessions were characterised by cyclical periods of intensive discussion and the trying out of ideas followed by episodes in which the experimentation became more routine. The variations in the nature of the activity meant that some aspects of the session were clearly more significant than others. The important issue seemed to be to maintain the continuity of the story and this was more easily done when the action was less intense by discursive accounts of what happened. These accounts aimed at giving an account of what happened with as little interpretation as possible.

The partially transcribed accounts of the clinical interviews were then brought into juxtaposition with the transcribed interviews. In the case of the final iteration, interpretative analyses were then inferred. These represented accounts of how the construction of meanings seemed to develop and the changes that may be attributable to the influence of the computer-based tools, the discussions with another child and my own interventions. This more detailed analysis of the clinical interviews seemed more important in the final iteration since justifications were now being sought for the issues which had emerged from earlier explorations.

The process of distilling the case analyses from the raw case accounts and pre-interview transcripts, and then further distilling the main issues, raises a complex problem of how to disseminate those issues without, on the one hand losing the sense of continuity in the unfolding of the children's meanings, whilst on the other hand, avoiding enormous amounts of repetition and complexity across all the different cases. Analysis of the eight case studies in Iteration 3 allowed me to arrive at the post hoc decision to focus on one pair of children, Anne and Rebecca, to illustrate many of the issues. Other case analyses would be used to illustrate significant variations from the case of Anne and Rebecca, whilst all the transcripts, case accounts and case analyses would be made available on the World Wide Web. In summary, this approach to dissemination is based on the following premises.

- It avoids repetition from one case study to another.
- It allows a focus on the main issues.
- It maintains a sense of continuity in the story of how the main issues unfold.
- The choice of Anne and Rebecca is not arbitrary but builds on the post hoc recognition that this one case covers many of the issues identified across all eight case studies.
- The brief reporting of other cases allows the dissemination of variations on the case of Anne and Rebecca.

4.6.3. The Development of Case Studies

The development of case accounts (Burgess, 1984) of individual or working pairs of children has been used to narrate the story of how the construction of meanings developed from the starting points apparent during the semi-structured interviews. The initial case accounts were then further developed through interpretative case analyses of what happened, an approach influenced by Mason's notion of giving an *account of* and an *account for* (Davis, 1992; Mason, 1994). The case analysis became the data, from which were distilled issues common to different case studies, or which varied in nature from one case analysis to another. The case analyses can also be seen as a research output in themselves as they will for many readers throw light upon their understanding of children's construction of meanings for stochastic processes.

Whether used as a starting point for further analysis or as an output in their own right, the case analyses raised an important question about how we might regard them as in any sense typical of a wider population. What are these accounts cases of?

The use of case studies in ethnographic research has provoked some debate on this issue. It seems that there has been a need to reconceptualise just what we mean by generalisability. One thing is clear; we can not claim generalisability in the sense that statisticians intend by the term.

The case study can offer meaning in a way that conventional studies can not (Donmoyer, 1990). It offers insights into areas which would not otherwise be accessible. In the case of this study, few people are in the fortunate position to be able to study young children who have the technological expertise of these young children. The case study can offer a view through the researcher's eyes. My experience of designing the computer-based tools, as well as working with the children using those tools, gives me a unique view, which can not be replicated and yet can form the basis of a meaningful story. In addition, the case study approach can reduce resistance on the part of the children. The engagement with the software was immediately evident and any barriers between the children and myself quickly disappeared. The task took over from any inhibitions that would, I am sure, have been very evident in a conventional testing paradigm.

Given these imperatives for using case-study approaches, how should we conceptualise generalisability? The roots of the case study approach lie in anthropology, where the aim is rather to seek out diversity and richness. Common attributes were seen as less interesting than differences. Nevertheless, case study approaches have become sufficiently widespread that the issue has become important in the way case studies are now used.

Generalisability needs to be reconceptualised as "fittingness", or "translatability" and "comparability", or "naturalistic generalisation". By giving thick descriptions of the situation observed, it is possible to determine intuitively whether the description fits another situation or not (Schofeld, 1990). As an output, the case studies will resonate (or not) with experiences which the reader recognises as similar.

Generalisability lies in that resonance. As a source of data for further analysis, the case studies will offer certain common threads and consequently there will appear to be some generalisability across the case studies. Those common issues as reported in this study may then themselves appear *fitting* to some readers. Where there is variation between the case studies, such diversity will not be devalued but recognised, as in the anthropological tradition, as something of value. Those of us who have worked with children for many years will in any case recognise how such diversity is in fact ubiquitous.

The tool use phase will be picked up in later chapters but the next chapter describes the development of tools, concluding with a range of heuristics and principles which are seen to underlie the Chance-Maker microworld.

4.7. SUMMARY OF CHAPTER FOUR

This chapter has set out in some detail the method by which the aims of this study will be pursued. Using considerations of epistemology and pedagogy as well as some aspects of the research literature, the study will begin with a first shot at what a few Boxer-based tools might look like. This primitive version of tool development will be used as a basis for the first iteration of tool use with children.

By a process of interviewing children prior to the use of the tools and then video-taping their interactions with those tools, insights may be gained as to how those tools might be developed. And so the next iteration begins with another phase of tool development.

As the iterative process progresses, the emphasis changed from one in which most effort was placed upon the development and modification of Boxer-based tools to the study of children using those tools. Eventually, convergence in the iterative design process was apparent in the need for little change to the tools.

At this point, the methodology involves taking a very close and systematic look at how the construction of meanings for stochastic phenomena is shaped by the Boxer tools and structures.

The next chapter reports in more detail the evolution of the early iterations of the study.

NOTES

15 There have been many such examples where I have been involved in the development and use of Logo microworlds as a teacher and as a curriculum developer. One such case was the development of a microworld called Newton, which encouraged and framed exploration of the movement of turtles acting under the influence of Newton's Laws. It turns out that the work with children operating in the stochastic domain has many parallels with that of older students using Newton. These parallels must surely have informed this early bootstrapping stage of tool development, at least at an intuitive level.

16 I have many memories of the frustration, teaching secondary school age children the mysteries of probability without the sorts of resources developed in this research.

17 In fact, at the beginning of the research, the school included children of age 12 but the school was restructured during the period of the research.

18 Not only is the history of software development very recent but the use of such tools in education has only been a serious enterprise in the last 30 years.

19 It is hard to understand now why this was the case. There seemed to be a breakdown in the epistemological analysis as it now seems so obvious that this tool should have been available from the outset. Hindsight is of course very powerful.

20 All names of the children are pseudonyms.

CHAPTER FIVE

The Iterative Evolution of a Domain of Stochastic Abstraction

5.1. OVERVIEW

This chapter outlines the interleaved phases of tool development and tool use through Iterations 0 to 3. This chronological description summarises the initial bootstrapping, exploratory, developmental and analytical stages. The software design issues, which unfold throughout this chapter, are encapsulated in the final version of the microworld, analysed and reported in Chapter Six, and the children's meanings, which emerge in this chapter, are studied more systematically in Iteration 3 and reported in Chapters Seven to Nine.

Chapter Five aims to tell the story of how the designing of software tools responded to the children's actions by attempting to resolve difficulties or exploit opportunities of a pedagogic or epistemological nature, illuminated by that activity. At the same time, a parallel story unfolds of how the children themselves responded to the tools and resources at their disposal.

The approach of this chapter will be to summarise the tool development and tool use phases of each iteration in turn; at each stage, I focus on the epistemological and pedagogic intentions embedded in the software tools and how the children's activity illuminated those intentions in new ways, resulting in the emergence of issues which then become the focus at each stage for the next iteration.

5.2. INTRODUCTION

The aims of this thesis, as set out in Chapter Three, are quite clear. This was not the case at the outset of this research. It is fair to say that the study has involved a continuous re-negotiation of the aims, resulting in that post hoc clarity. Perhaps it is inevitable that qualitative illuminative research will involve some reassessment during the research of what precisely is being studied. What is special in this case is that this process has been made explicit by the continuous and iterative programming of my intentions into a computational medium. There are 140 sub-versions of the microworld, supported by diary notes maintained as the software developed, which stand as testimony to the minor reassessments and occasional major re-thinks. Indeed, the original software tools were inspired by a separate study of children's active graphing (Pratt, 1995), in which meanings for scatter graphs emerge out of their immediate and ongoing use during the collection of data in experiments. Aiming to build on this prior work, I was searching for new contexts for active graphing, and was considering the graphing of stochastic processes. During Iteration 0, the focus began to shift from graphing to the children's meanings for the stochastic. Nevertheless, it is not difficult to trace elements of the interest in active graphing as I look back at the early versions of the software.

5.3. ITERATION 0 : BOOTSTRAPPING

5.3.1. Tool Development

I wish to begin by considering briefly how the focus on the stochastic developed out of an interest in active graphing. In classical active graphing, a scatter graph has the potential to help a child to make decisions about an experiment. In stochastic 'experiments', such as surveys, the child does not so much control an independent variable as collect sets of data relating to members of a sample. In this context, graphing has little utility *during* the experiment. Any purpose for graphing in such surveys is not realised until the presentation of results, usually at the end of the project. In searching for a way of applying active graphing to the stochastic, I recognised the need to give the child some sort of control over the stochastic model. Perhaps I could invent an activity in which the underlying model was probabilistic in nature and yet was accessible to, and manipulable by, children.

Such an approach might lend utility to graphing in a stochastic context, but raises issues about why a child should wish to engage with the stochastic model in the first place. The activity would need to be seen as purposeful by the child. The notion of computer as design instrument seemed to be one to which young children could relate (especially children from the Primary Laptop Project, who were sophisticated computer-users).

After considering various possible products of a design process, I settled on the idea of the children making a game, similar to that commonly found on stalls at fair grounds, in which a coin is rolled down a gentle slope. The player loses (and so the stall-holder wins) if the coin lands on one of the lines drawn at equal spaces across the board. The design process would involve deciding on the size of the board, the width and the spacing of the lines, in order to make the game attractive to potential customers and yet yield a profit.

The roll-a-penny activity seemed at face value to be intrinsically interesting to children, offering a design context in which the role of the computer would be transparently sensible, whilst at the same time embedding a stochastic mathematical model. Early versions of the software, with their emphasis on graph plotting, tended to present cluttered screens and seemed to leave the child with too few ways of engaging with the tools, which operated autonomously for much of the time. As this clutter was gradually removed, the central importance of graph plotting was diminished.

The tools used by the first pair of children in Iteration 0 were embedded in Version 0.26 (Figure 5.1).

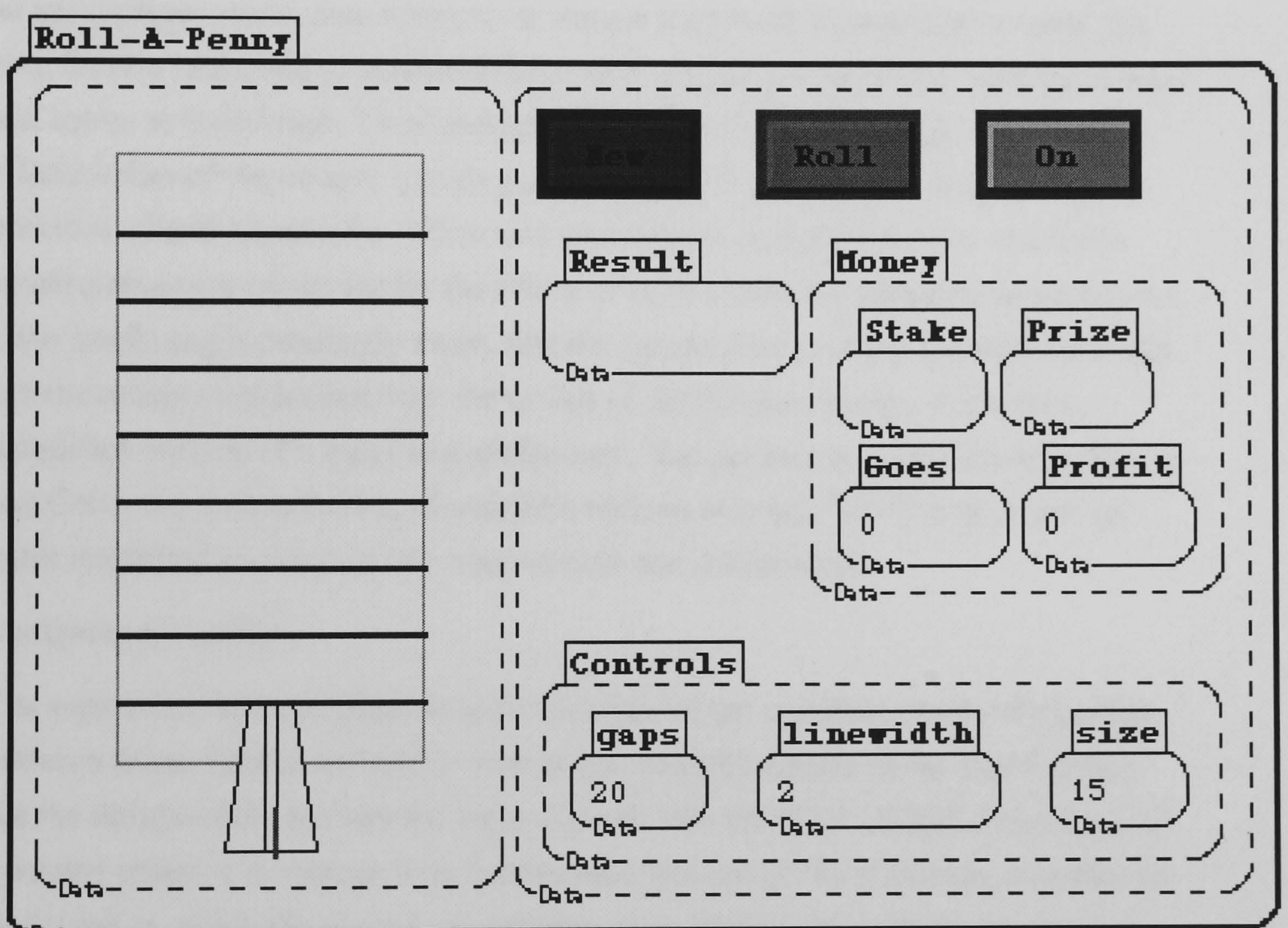

Fig. 5.1 : The tools in Version 0.26

Figure 5.1 depicts the roll-a-penny board. The focus on profit remains, as illustrated by the *money* box, but the graph plotter has disappeared into the background. By clicking the *roll* button, a coin could be made to travel up the board, coming to rest at a random distance from its starting point. A child could vary the stake and the prize in order to explore the profit over a number of games. Control of the size of the gaps between the lines, the width of the lines and the size of the coin could be exerted by editing the corresponding boxes. In this way, the design of the board could be altered, impacting on the size of the profit.

5.3.2. Issues Emerging from Iteration 0

In Iteration 0, just one pair of children, Sandra and Colin, tried out the software merely to offer me a first impression of how young children might react to such Boxer-based tools. Through the phases of tool development and tool use, the following issues emerged and would inform tool development in Iteration 1.

Designing for purpose

For Colin and Sandra there was a clear purpose in using the software tools, which was to identify whether the default design of the roll-a-penny board would generate an appropriate profit, and, if not, to re-design the board. In working towards this aim, Sandra and Colin generated batches of five trials and noted the number of wins and losses in each batch. They averaged the number of wins overall to judge the effectiveness of the design, applying a utility for average already abstracted from previous school experience. There was no evidence though of Sandra and Colin constructing new meanings for the utility of other stochastic concepts. In particular, I was becoming increasingly aware that the current design of the software tools did not encourage engagement with the notion of distribution, except in the very simplified version of a win / lose dichotomy. The purpose as constructed by Sandra and Colin did lead to the use of concepts such as average, but by now, I was far more interested in meanings for randomness and distribution.

Designing for utility

The separation between purpose and utility helped me to reflect on the relationship between them. Sandra and Colin were able to control aspects of the board design but the design of the activity led them to work with the WIN / LOSE dichotomy. If I wanted children to engage with fundamental notions of the stochastic, I needed to find ways in which the control mechanisms were likely to bring them into contact with those concepts. For example, if Sandra and Colin had been somehow required to manipulate the distribution of the distance travelled by the coin, perhaps the concept of distribution would take on new meanings for them.

Manipulating access

The process of developing tools and then removing or hiding them had made me particularly aware of one aspect of the spatial metaphor in Boxer. Boxes can be made more or less accessible by how they are positioned in the environment. Boxer possesses at least four levels of such structuring, which in order of decreasing accessibility are:

(i) open boxes available directly at top level,

(ii) shaded boxes available directly at top level,

(iii) closed boxes available directly at top level,

(iv) boxes hidden on flip-sides of other boxes,

(v) boxes hidden in closets.

By moving the position of a box, I could influence the likelihood that a child would engage with its content. For example, the graph plotter, which had been unavoidable in the very early versions of the software was now hidden in a closet, on a flip-side, only really accessible through intervention, should the children's actions suggest that it would be useful. More generally, it was clear that the screen design would focus attention towards features made prominent and away from hidden or disguised features.

Enabling the recording of results

In discussions with Sandra and Colin after the tool use session, they suggested that ways of recording results would have been helpful. The plotter, which had been relegated to the background, would have served this function, but it was now clear to me that I needed to find more straightforward ways for children to look back at their results.

Such a step would, according to Sandra and Colin, have functional purpose but, at the same time, the introduction of such a tool might act as a window on how children made use of historical results to make predictions about the future. With these issues in mind, I entered into the next iteration of tool development.

5.4. ITERATION 1 : EXPLORATORY

5.4.1. Tool Development

I began to explore ways of re-conceiving the computer-based task. The fundamental change involved exploring ways of switching from an emphasis on game design so that the child would need to work with more complex issues of distribution. My attention was drawn to one particular distribution, which currently was handled by the computer, but which could become a control point within the roll-a-penny context. I was thinking about the distance travelled by the coin. From a mathematically informed perspective, we might construct the distance travelled as a Normal distribution, with most coins travelling close to a mean distance, and only rarely travelling much further or much less than that average distance. One focus might be to study how children think about the distance rolled by a coin, and how computer-based resources might shape that thinking. The tools that were developed in response to this new focus are depicted in Figure 5.2, which illustrates Version 1.13 of the software.

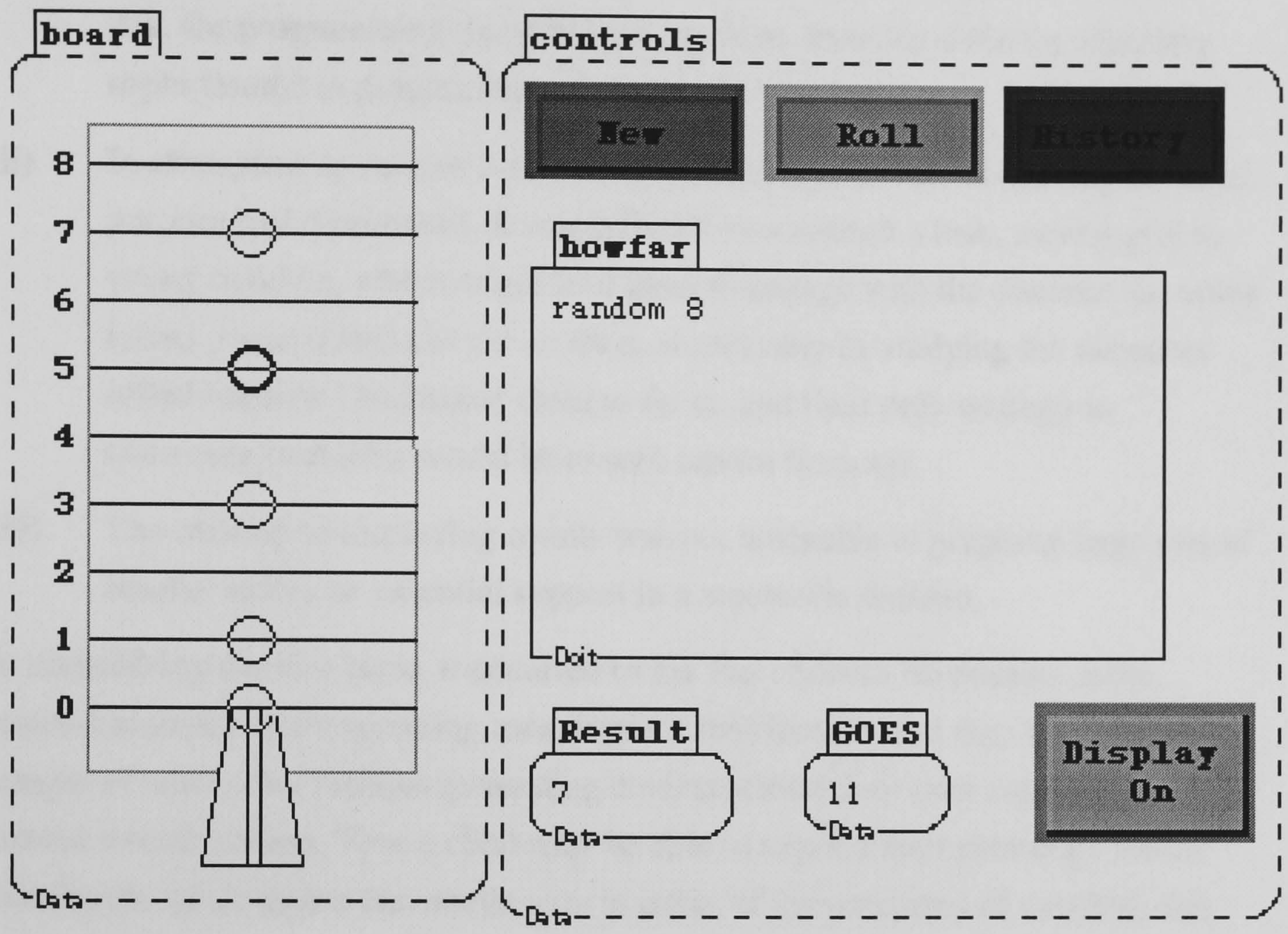

Fig. 5.2 : The tools in Version 1.13

The change in the nature of the activity is demonstrated by the removal of the money box and the central nature of the *howfar* doit[21] box. It was envisaged that children would express their thinking about the distribution of distances rolled by programming the howfar box.

In the diagram, the howfar box is shown in its intended default form, where the coins would roll according to a uniform distribution.

The controls of the board design have disappeared, since I no longer wished to place high priority on design issues. The child is now able to inspect the result box which would contain the distance rolled by the coin. It now became important to be able to look back over past results enabling a comparison between mental images of the distribution and the actual picture of results. The coin's resting places could be re-displayed by pressing the *history* button, as illustrated in Figure 5.2.

There were some clear pedagogic and technical difficulties, which needed to be addressed before moving onto a new phase of tool use.

(i) Although some progress had been made in terms of amending the controls so that the child would engage with concepts such as distribution, the howfar box seemed too open. It was difficult to imagine any initial meanings that children might bring to coin rolling. Perhaps more serious still, the programming threshold felt too high, even for children relatively sophisticated in programming.

(ii) In attempting to resolve issues to do with utility, it seemed that any sense of purpose had diminished. It was difficult to construct a task, meaningful to young children, which might lead them to engage with the distance the coins rolled. I was afraid that the children would only be studying the distances rolled because I had asked them to do so, and their only strategy to overcome problems would be to seek advice from me

(iii) The method of displaying results was not amenable to graphing large sets of results, surely an essential support in a stochastic domain.

In considering the first issue, it occurred to me that children do possess some internal resources for expressing meanings for distribution, and they lie in mental images of what other random generating devices within their own cultures are similar to coin rolling. Thus a child may be able to express their meanings for the distribution of distances that a coin rolls in terms of the outcomes of a rolling dice or a lottery draw.

I therefore began developing tools, which I saw as aids to expression. The model in my mind was that the children would think of themselves as using these tools in an experiment which would control the roll-a-penny coin. The children had, for example, a tool, called *dice*, and a doit box called **roll**; the children set up an experiment, **roll dice**. The computer determined how far the coin would travel as if by the roll of a dice.

I considered a range of random generators, evoking images of childhood in an attempt to think of tools familiar in their everyday lives. Coins, lotteries, dice, wheels of fortune, playing cards were all images that sprung to mind. I also introduced the donkey-tail tool. The donkey-tail tool was based on the party game in which a blindfolded child tries to pin a tail onto a picture of a donkey. These ideas were incorporated into a new version of the software, depicted in Version 2.09 (Figure 5.3).

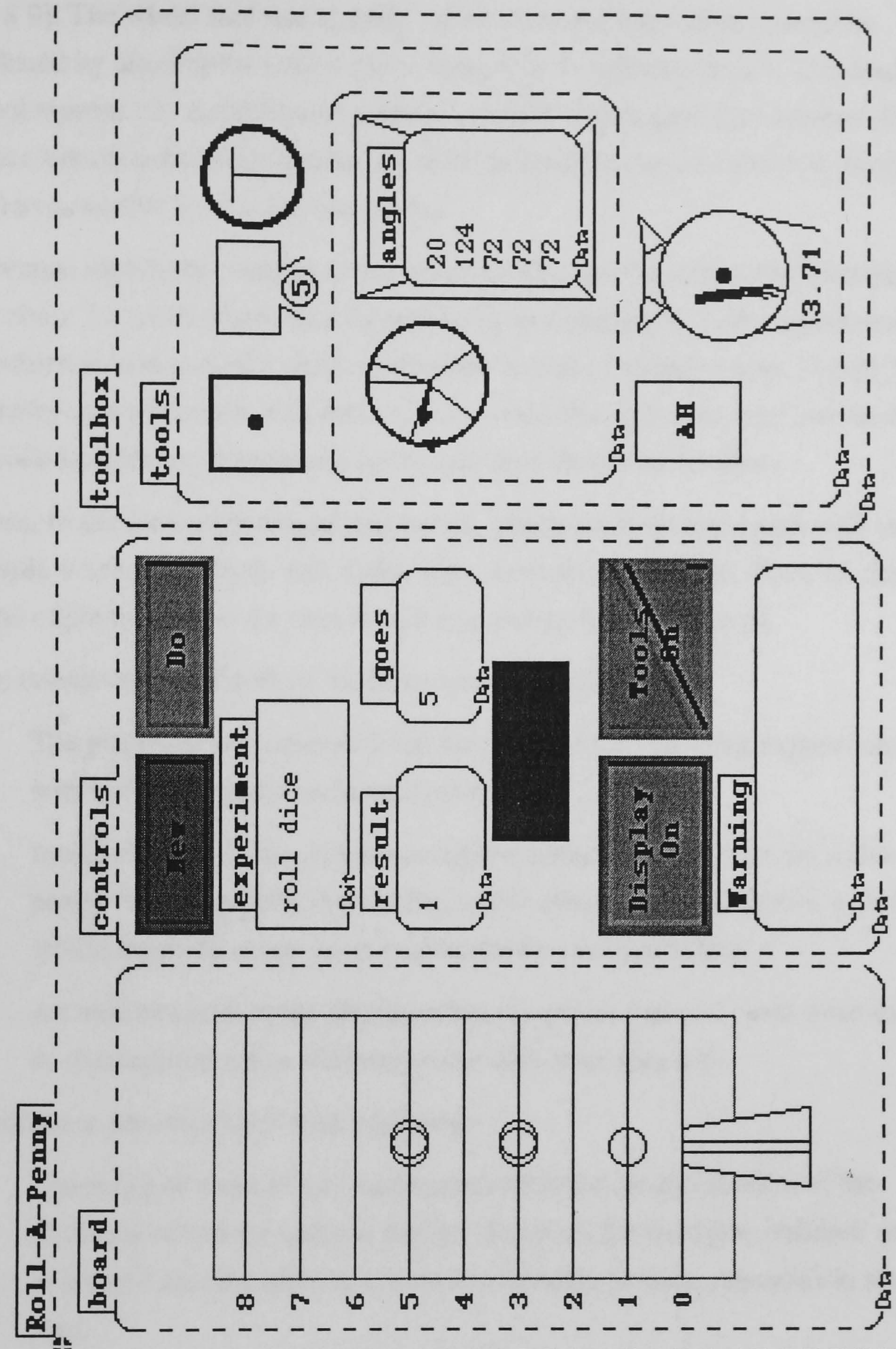

Fig. 5.3 : The tools in Version 2.09

Each of the tools displayed a result when activated by a mouse-click. The dice, coin, cards and lottery tools were similar in that each, by default, represented a random generator with a number of equally likely discrete outcomes. For example, the dice tool was a data box containing the numbers [1 2 3 4 5 6]. The contents of any of the tools could be edited. For example, the contents of the dice tool could be altered by the child. So bias could be introduced by changing the dice box to contain, say, [1 2 3 4 5 6 6 6] or a nine-sided dice might be represented as [1 2 3 4

5 6 7 8 9]. The wheel tool was slightly different in that bias could instead be introduced by altering the angles of the sectors in the adjacent box[22]. The donkey-tail tool represented a continuous random variable, which gave the outcome (the distance between the random position of the tail and the point at which it should be fixed) as a number to two decimal places.

The central idea in this version of the software was that the child expressed their ideas about the roll-a-penny tool by setting up an experiment, in the *experiment* box, which would generate similar outcomes to that of rolling a coin. Figure 5.2 illustrates an experiment, **roll dice**, and so, when the *do* button was activated, the coin rolled a distance determined by the outcome shown on the dice.

Actions, in the form of prepared doit boxes, which could be combined with any of the tools were: **roll, toss, cut, spin, pin, nextof** and **shuffle**. Each of these actions might be used in the experiment box alongside a named tool.

Let us review each of the three modifications listed above:

(i) The programming threshold had been reduced by allowing expression in terms of ready-made tools and actions.

(ii) It was envisaged that children would be asked to try to make the roll-a-penny 'work properly' by linking it with other appropriate tools, a challenge which might be taken on as purposeful by young children.

(iii) An improvement to the display of results meant that outcomes were shown as overlapping coins, allowing more data to be graphed.

The software was still beset with problems:

(i) The range of ways of expressing meanings for the distribution of the distances rolled seemed too limited. Suppose, for example, children wanted to suggest that the distances were not random, perhaps patterned in some way.

(ii) There was a complexity about the mixture of actions and tools, which could be very confusing.

(iii) The method of displaying results was still limited.

Attempts to work on each of these areas of difficulty resulted in Version 3.21 of the software (see Figure 5.4), which emerged not only as a solution to the problems but also as the exploitation of an opportunity.

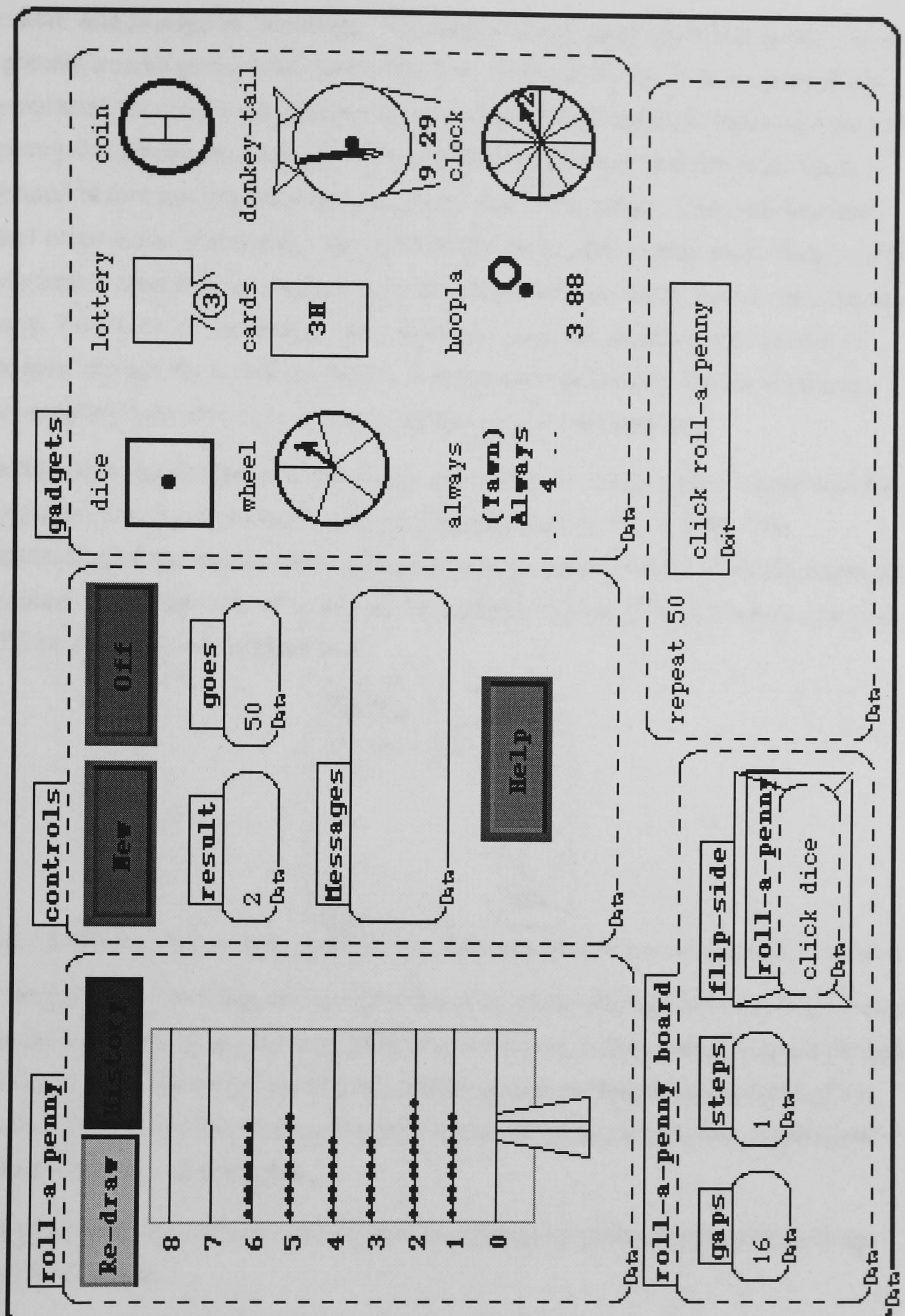

Fig. 5.4 : The tools in Version 3.21

It had become apparent that the tool paradigm offered a new possibility; the roll-a-penny simulation became just another gadget on a par with the others. The software design had become a task of making the operation of roll-a-penny and the other tools as similar as possible. The roll-a-penny tool was presented to the children as 'broken' and in need of 'mending'. The notion of mending the roll-a-penny seemed to present a more purposeful task to children whilst at the same time opening up possibilities for observing children's expression of mathematical ideas on how roll-a-penny should operate. The child first needed to discover how the other tools operated before making links between them and roll-a-penny. The tools became rather more autonomous since the child needed to be able to play with them in order to construct meanings for their behaviour, before they could be used to mend roll-a-penny. To reflect an increase in their independence, the tools became known as 'gadgets' though the notion of gadget developed considerably in later iterations. Roll-a-penny was also now seen as a gadget, albeit a broken one.

Linking between the gadgets was extremely important. Each gadget could now be flipped to reveal a command such as **choose-item [1 2 3 4 5 6]**. The introduction of the **choose-item** primitive expressed the notion of randomness and appeared on the flip-side of nearly all the gadgets (Figure 5.5). As before, the data could be changed to introduce bias.

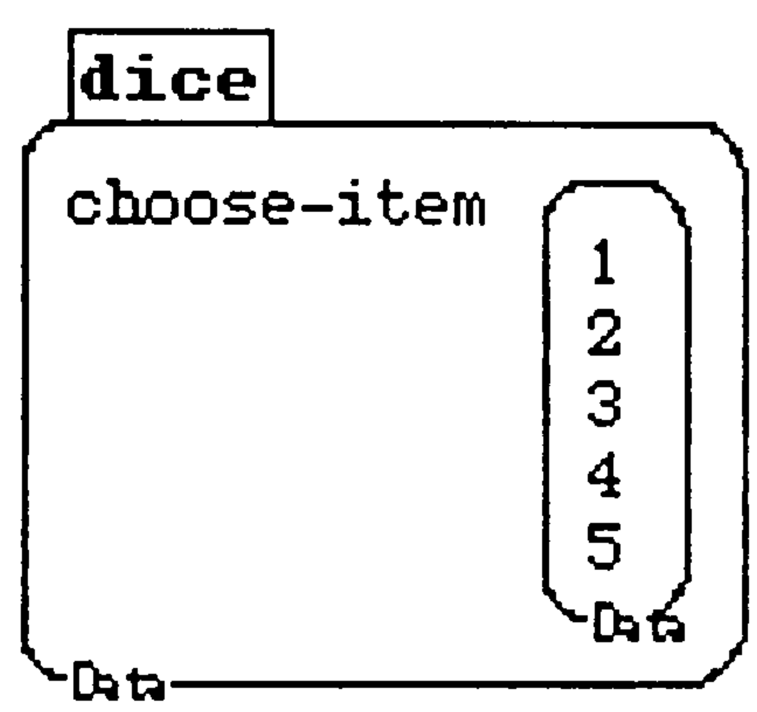

Fig. 5.5 : Each gadget could be flipped to reveal a command such as choose-item [1 2 3 4 5 6]

All the gadgets, when flipped, revealed the core mathematical idea of how it worked with the rest of the programming hidden in the closet. In this way, it was envisaged that the child would be helped to focus on the central mathematical element of the process. The parallel between roll-a-penny and the other gadgets was emphasised by their similarity of operation.

This version of the software can be seen as addressing each of the previous three areas of difficulty.

(i) The collection of tools included three innovations. The *hoopla* represented the fair- ground game in which a player attempts to throw a hoop around a target peg. The hoopla reported the distance between the hoop and the peg. The hoopla was seen as supportive of the donkey-tail tool in that it represented a continuous distribution The *clock* was introduced as a tool for expressing the notion that the roll-a-penny results were patterned. Children could also express a belief that the coin might constantly roll the same distance through the *always* tool.

(ii) The **click** primitive replaced all the previous actions by drawing on a feature common to all the tools — a tool was activated by a mouse-click on the tool. Instead of drawing directly on our language of how we use these tools in everyday settings, I called upon their recent experience of how the tools were activated in the software.

It was now possible to activate the dice, or any of the tools, by a direct mouse-click as before, or by execution of a command such as **click dice**. The roll-a-penny tool could be similarly activated by a command **click roll-a-penny**. The simplification of removing all the different actions was a considerable step forward. The tools became interesting to explore in their own rights, through commands such as **repeat 50 [click dice]**.

The introduction of the **click** primitive made it much easier to express repeated experiments. In order to encourage children towards the carrying out of lots of trials, I placed the **repeat** primitive at top level on the screen.

The **click** primitive also encouraged linking between gadgets. The roll-a-penny gadget was set by default to **click always**, representing a constant outcome. The roll-a-penny gadget could be mended by editing its flip-side to use the **click** primitive (e.g. **click dice**, if it was felt that the behaviour of the dice adequately represented the behaviour of the rolling coin), or the **choose-item** primitive (e.g. **choose-item [1 2 2 3 3 3 4 4 5]**).

(iii) The method of graphing previous results became more conventional. The distances rolled were represented in pictogram form, as shown in Figure 5.4.

The tool development then in Iteration 1 culminated in the creation of a set of gadgets, quasi-stochastic computational mini-systems or devices, which could be used as stand-alone objects but which could also be used as descriptors for other objects in the domain. I now turn my attention briefly to how these tools were used by children.

5.4.2. Tool Use

Version 3.21 of the software was used by two pairs of children in Iteration 1. For illustrative purposes, I wish to concentrate on one two hour session with Jenny and Gill[23], both age 10. The quotations below include discursive narrative, which aims to give a plain and uninterpreted account, and direct transcriptions from the videotape

At the outset, the two girls were encouraged to play with the gadgets. Gill and Jenny soon recognised the random nature of the dice, lottery, cards, and coin gadgets. However, they seemed to demonstrate a different perspective when exploring some of the other gadgets. The following extract began with Gill and Jenny already using the hoopla gadget. It was clear that they expected to be able to control in some way where the hoop landed.

> Gill asks, "Why doesn't it go to the places that you point at?" I clarify, "That's what you feel should happen, is it?" Both girls respond, "Yes." I ask, "When you click the dice, do you expect ... do you expect the number that comes up on the dice to be anything to do with how you click the mouse?" Gill replies, "It won't really but it could be." I ask, "OK. What about the wheel? Would you expect the number that comes up on the wheel to be anything to do with how you click the mouse? Gill: "Well, the mouse? It might depend on what you mean .... actually clicking the mouse or placing the mouse?" I explain, "Yes, I'm including placing the mouse as part of ...." Gill answers, "Yes, it does, I think." When I ask Jenny, she says, "Not really." I continue, "So you don't, but Gill does. Why do you think it should, Gill?" Gill says, "Well cos every time I clicked there it always came there." I suggest, "Did it? Do you want to try again?" Gill tries to click on the wheel but the spinner shows a different place. Gill exclaims, "Oh, it did before." After a few more clicks, I ask, "What do you think now?" Gill says, "I don't know — it might not."

There were similar episodes concerning the hoopla and the donkey-tail gadget; in each case the children had expected to be able to control the outcome by how they clicked the mouse but further careful experimentation showed that this did not seem to be the case. The girls began with a perspective in which the wheel, hoopla and donkey-tail gadgets should be controllable (and were quite different from the dice, lottery, cards and coin gadgets). But experience proved otherwise; however they experimented, they were unable to make the gadgets respond causally to any action.

Later in the session, I introduced the roll-a-penny gadget. After some initial exploration, Gill and Jenny were challenged to make the roll-a-penny gadget work as realistically as possible, that is, to simulate the rolls of a real penny. In a previous session, the girls had played with a real roll-a-penny device and observed how the distances that a coin rolled varied slightly even when they tried to keep all the possible controlling factors the same (see Appendix A1.1 for a picture of the real roll-a-penny apparatus and the interview schedule in which it was used). They had also discussed how most coins seemed to roll roughly the same distance although they noticed that a few went further and a few fell short.

The children began by modifying the dice data box, first to include just 4 and 5 and later adding 3 and 6. After 40 rolls, the children used the history button to obtain a picture which showed a roughly equal number of each score (Figure 5.6).

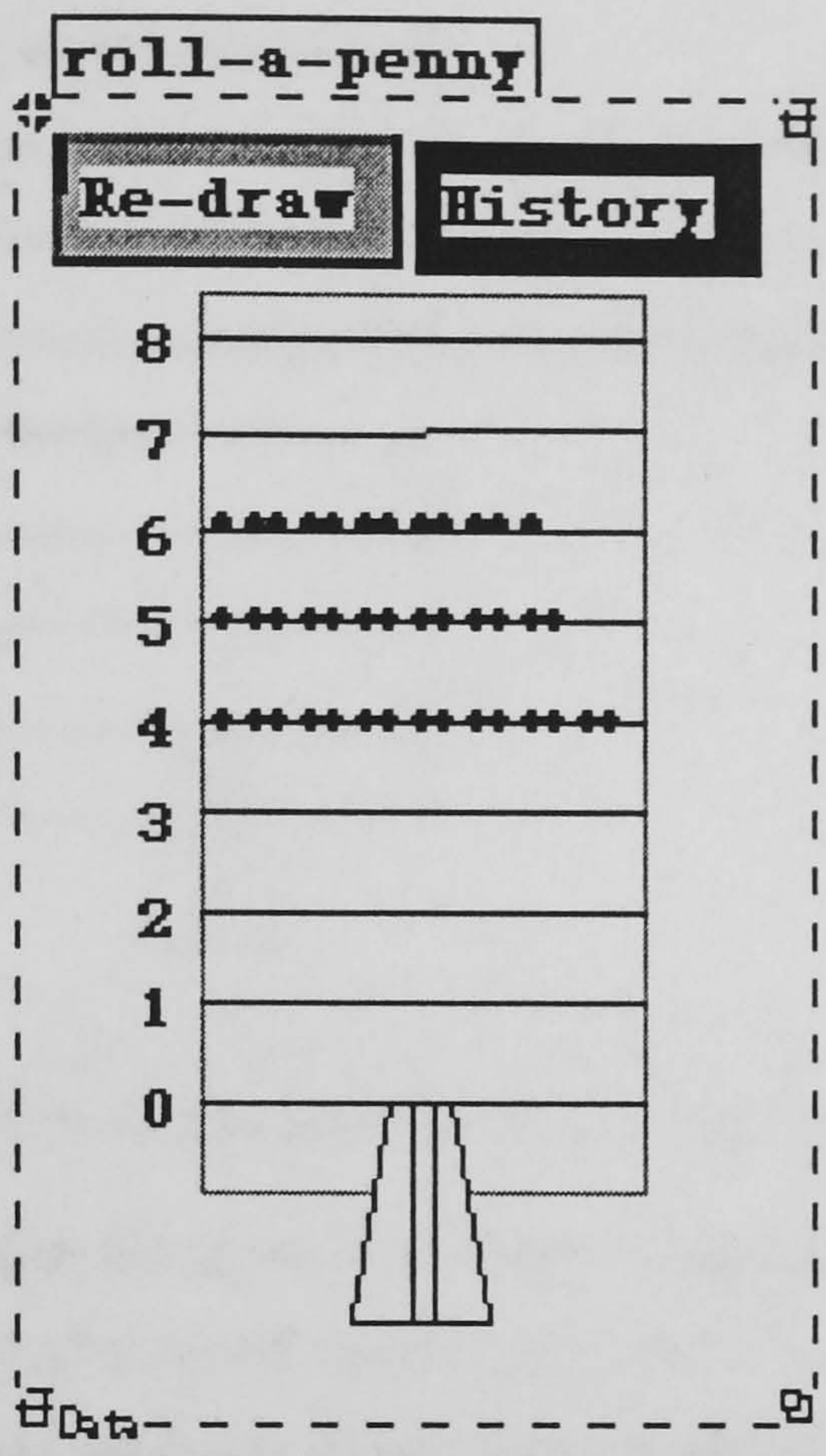

Fig. 5.6 : A picture showed a roughly equal number of each score

I asked Jenny and Gill how the picture on the computer compared to the picture previously obtained by marking the positions of the rolls a real coin.

> Gill replies, "Well, because we wanted the computer to look more realistic, if we put too many numbers on, it could have went anywhere, but if we do this, it's, I don't know, it's on its own, we can't control it........We might be able to control that if we only put one number and then they'd all be on the same number but that would be a bit stupid." I say, "Yes, and that would lose the realism of them going different distances." Gill adds, "Yes, and if we like put a six and a five, like, they'd all be on the same ..." I encouraged the girls to think about a real dice and its six faces, "So, you've got these six faces on the dice. What could you do to them to try and make it more like what you're doing here?" Gill replies, "You could put like more of the number on one, so, like you could take the 2 off and put 6 on it instead, and you could take the 3 away and put a 5 on."

They then began to change the dice on the computer. Initially, they made the data box for the dice gadget into [4 5 6 4 5 6 7 8 3]. After several more changes, they reached a point where they had four of each of 4, 5 and 6, one 7 and one 3. When they rolled the penny lots of times, they obtained a picture with many more 4's, 5's and 6's and rather less 3's and 7's (Figure 5.7).

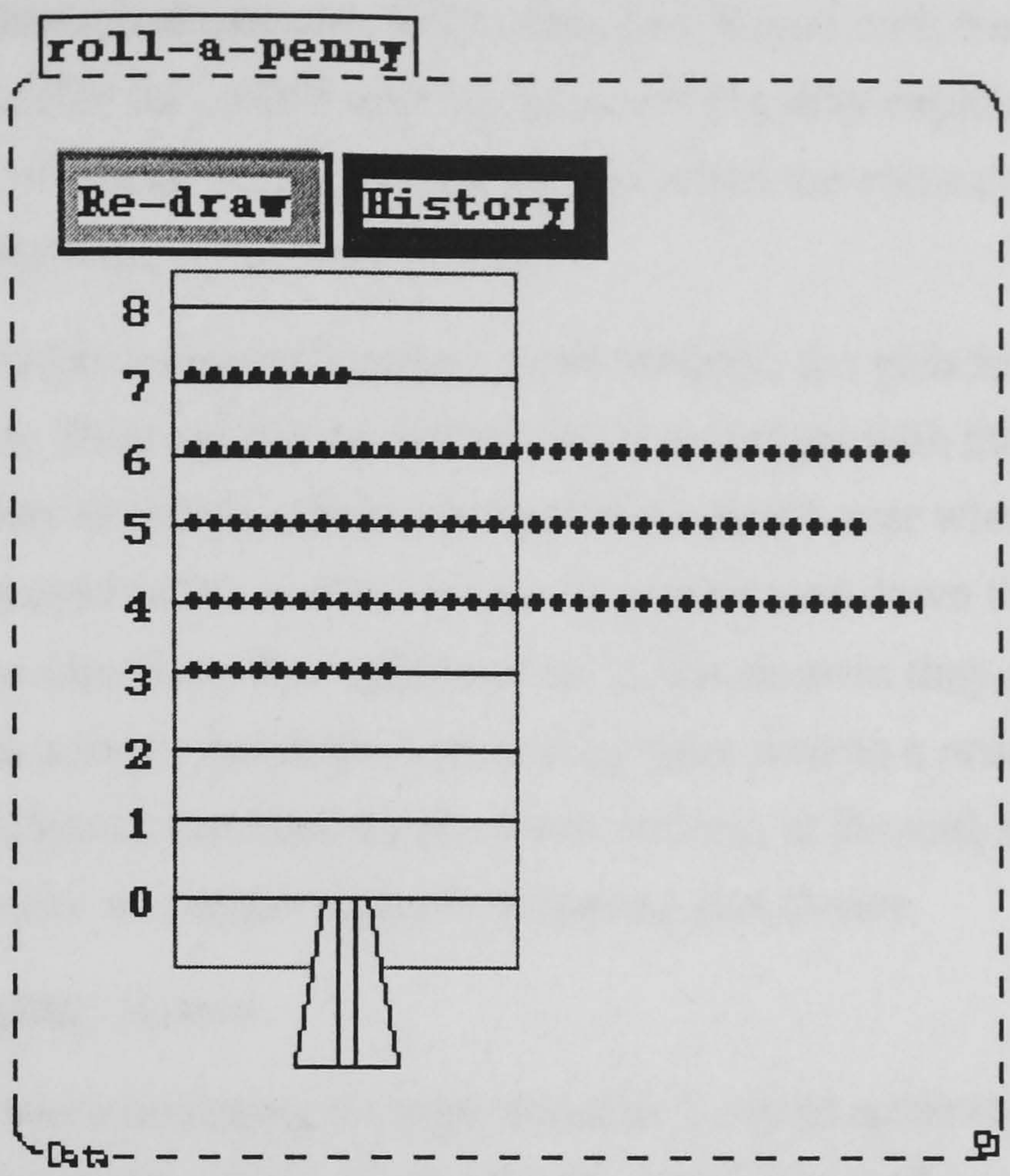

Fig. 5.7 : They obtained a picture with many more 4's, 5's and 6's and rather less 3's and 7's

Jenny and Gill explained that there were so few 7's and 3's because they had only one of each whereas they had four of the 5's, 6's and 7's. They felt that this picture was much closer to the real results that they had previously obtained. Finally I asked Gill and Jenny about the extent to which they could control the real world coin and that on the computer.

> Gill says, "Well it's a bit easier to control on the computer ...if you go back onto the dice thingy, you know on the back of it, it will tell you what the numbers are and you can take them from it, say like, if you wanted them all to be 6's, you can just write 6 on it and it will always come out with a 6." I ask, "Right, so that's a sort of control, isn't it? And what control do you have on the real roll-a-penny?" Jenny says, "None, unless you push it or something."

Gill and Jenny came to this domain expecting that they would be able to control certain gadgets and not others. They were not surprised that gadgets like the dice behaved stochastically. They knew from their everyday lives that such devices were unpredictable and uncontrollable, and this seemed to be a major criteria by which they judged whether a situation was stochastic.

However, they expected to be able to control gadgets such as the wheel, hoopla and roll-a-penny. Their everyday experiences suggested that there were factors in their control, such as how hard they threw the hoop, which affected the distance travelled, and they were surprised when they found that, in this domain, they could not exert control through the mouse. When they had played with the real roll-a-penny device, the coins had rolled varying distances but they explained this variation in terms of *causal* factors, how they had rolled the coins or the effect of the slope down which the coins were rolling.

In seeking to make the computer's gadget more realistic, the girls looked to introduce variation. They did this by linking the dice gadget with the roll-a-penny gadget. Initially they found that there was too little control over where the coins landed when compared to the real device, so they narrowed down the options from which the dice was choosing. Recognising that in this domain they could easily introduce bias was a major breakthrough as they were now in a position to focus on *and control* the distances travelled by the coins. Indeed, at the end, Gill seemed to feel that the computer was easier to control than the real device.

5.4.3. Emerging issues

The issues which were emerging through Iteration 1 can be separated into notions of how the children constructed meaning for the stochastic, and issues related to the design of the software.

Meanings for the stochastic

The two pairs of children expressed two broad meanings for the behaviour of the gadgets.

- The children articulated a meaning, cued by the context, related to whether the gadget was controllable or not. Gill and Jenny had an expectation that the dice would be uncontrollable whereas they expected to be able to control the donkey tail. Similarly, for Gill and Jenny, the lottery gadget would be uncontrollable, whereas they expected to be able to control the hoopla.
- The children also articulated an initial meaning, related to the unpredictable nature of some gadgets, such as the dice. On the other hand, they expected to be able to cause particular outcomes on gadgets such as the hoopla. Thus unpredictability seemed to be a powerful discriminator of stochastic behaviour.

Key design principles

Some key principles were emerging from the observations of the two pairs of children in Iteration 1 which would inform the design of further iterations. These principles are outlined below:

- A key facet of the design lies in the expressibility of chance: that is, the notion of linked gadgets seemed to open up the possibility for learners to express that which is not causal.
- An important mechanism for this expression seemed to be in the way that the children edited the data on the flip-side of each gadget. Through such actions, they exposed their thinking to scrutiny, whilst simultaneously exploring the effect of such manipulations on the gadget's behaviour.
- Cultural expectations were evidently influencing the way the children responded to the gadgets. There was much evidence of children beginning with different expectations of those gadgets, such as hoopla and donkey-tail, which seemed to incorporate a games element, the suggestion that there was a goal to be achieved when activating the gadget.
- Various types of control were articulated during the tool use phase. On the one hand, the children expected to be able to influence, through direct mouse operations, the result for some gadgets, whereas later they began to exert control through manipulating the data in the flip-side of the gadgets.

The tool use phase then had clarified a number of opportunities, which changes to the design of the software might exploit, as well as some weaknesses in the current design.

(i) The gadgets, like the dice and the coin, were proving to be of interest in themselves in the way that the children were beginning to expose their meanings for randomness whilst engaging with them. There were opportunities here to review the role of these gadgets, making them more central in the overall design of the microworld and the children's activity.

(ii) The facility to change the way that any particular gadget operated was an important feature, which could perhaps be built into the gadget design in a more intuitive way.

(iii) The way that activity with a gadget seemed to be cued by cultural expectations suggested that there might be benefit in further emphasising the link between the gadget and its everyday equivalent.

(iv) The notion of control, or lack of control, was important. Perhaps a feeling of causal control could be incorporated into the software, allowing children to recognise the difference between physical control of the gadget and control through manipulating the possibility space. Typically, everyday encounters with the stochastic involve only physical controls. At present, the microworld only allowed control of the possibility space.

(v) As interaction through changes to the data on the flip-side of the gadgets became more important, I was increasingly aware of the complexity of the hoopla and donkey-tail gadgets. The source of this complexity lie in the need to represent sampling from a continuous distribution. At the same time, these gadgets had unintentionally cued feelings of goal-oriented games-playing, which at times seemed to hinder the observation of meanings for the stochastic.

5.5. ITERATION 2 : DEVELOPMENTAL

5.5.1. Tool Development

Iteration 1 had illuminated a variety of issues, which spawned considerable developments during Iteration 2. This phase of tool development marked some fundamental changes which continued through to the final version of the microworld.

The experience of observing children in the previous phase amending the way that certain gadgets operated inspired the thought that perhaps they could be engaged in an activity in which several of the gadgets were 'broken'. It might be proposed to children that they should try to program these gadgets to behave like their everyday counterparts. The task for the children could become one of identifying which gadgets were not working properly and then to mend them, using the tools provided. This re-conceptualisation of the software had two broad implications, deriving from the need for each gadget to be autonomous.

- Roll-a-penny would be relegated in importance, so that all gadgets would become equal in standing. This seemed, on reflection, to be the logical culmination of a process in which other tools had moved from being devices to express the behaviour of roll-a-penny to become the focus of attention in their own right.
- Whereas previously results were only stored in the case of trials of the roll-a-penny gadget, it would now need to be possible to record outcomes for all the gadgets. Therefore, the corresponding tools must be made accessible from all gadgets.

The process of incorporating these fundamental ideas into a microworld design culminated in Version 4.20 (Figure 5.8).

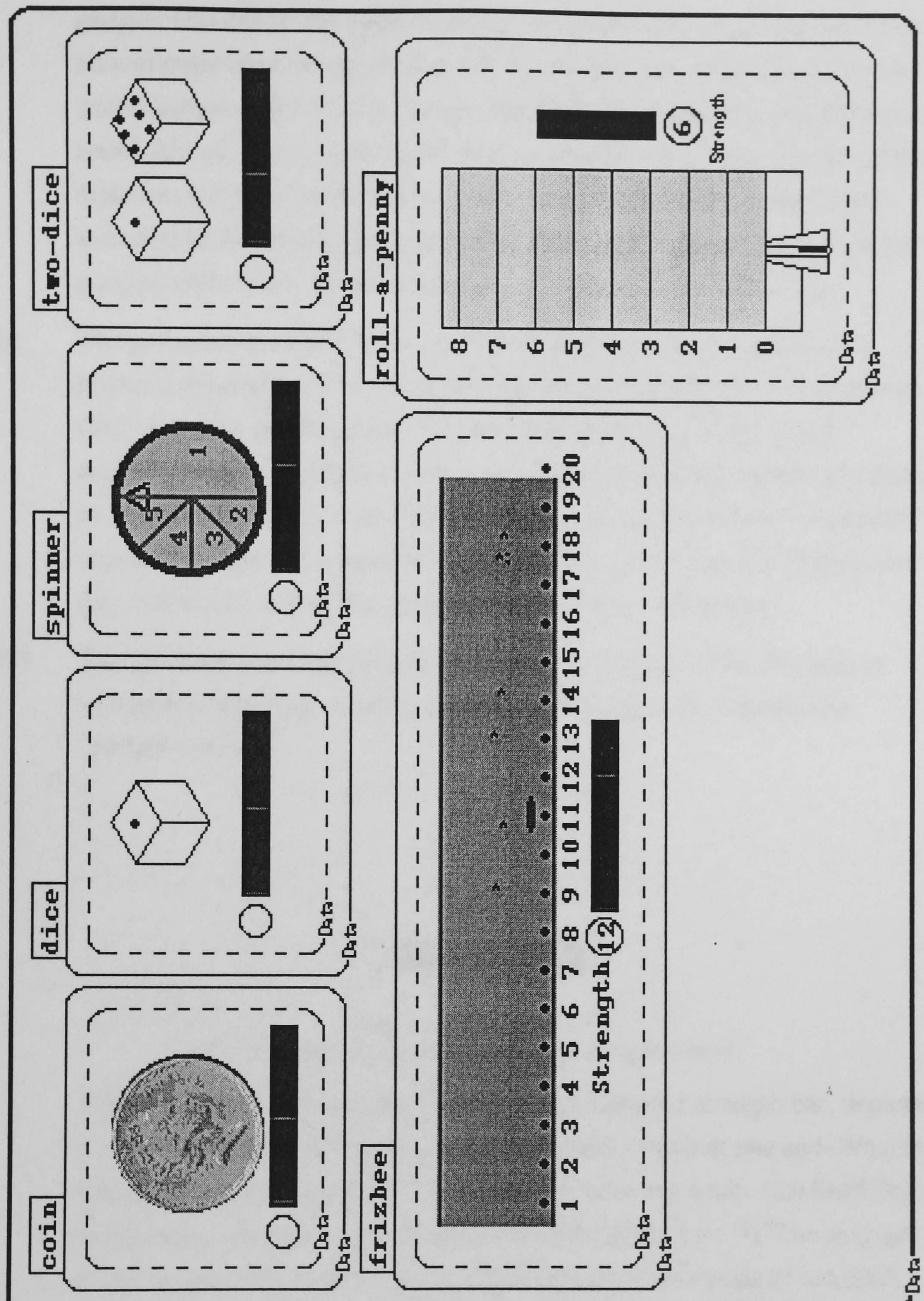

Fig. 5.8 : The tools in Version 4.20

This version of the software responded to the opportunities and weaknesses identified from the previous iteration in the following ways:

(i) The donkey-tail and hoopla gadgets were dropped. In their place, two new gadgets were introduced. The Frisbee gadget was based on the open air game of throwing a Frisbee. Its advantage over the donkey-tail and hoopla gadgets was that it was rather less goal-oriented in the sense that there was no particular target to achieve, and in that respect was more like the roll-a-penny gadget. The two-dice gadget was introduced since the distribution of totals, scored on two-dice, could be constructed as a discrete version of the distances travelled by the roll-a-penny coin. The two-dice gadget might therefore be interesting because of its links to the Frisbee and roll-a-penny gadgets without the added complexity of a continuous distribution.

(ii) The gadgets were made to be much more familiar in two respects. The physical appearance of the gadgets was now a much closer resemblance to their everyday doppelgängers. Each gadget, reflecting its increased importance in the microworld design, responded dynamically to activation. In previous iterations, only the roll-a-penny gadget had offered a dynamic image of its operation through the rolling coin. Now, the coin flipped, the dice rolled, the arm of the spinner spun, and the Frisbee flew.

(iii) The growing awareness of control as an issue had led to the decision to incorporate a feeling of causal control. Each gadget now contained a strength control.

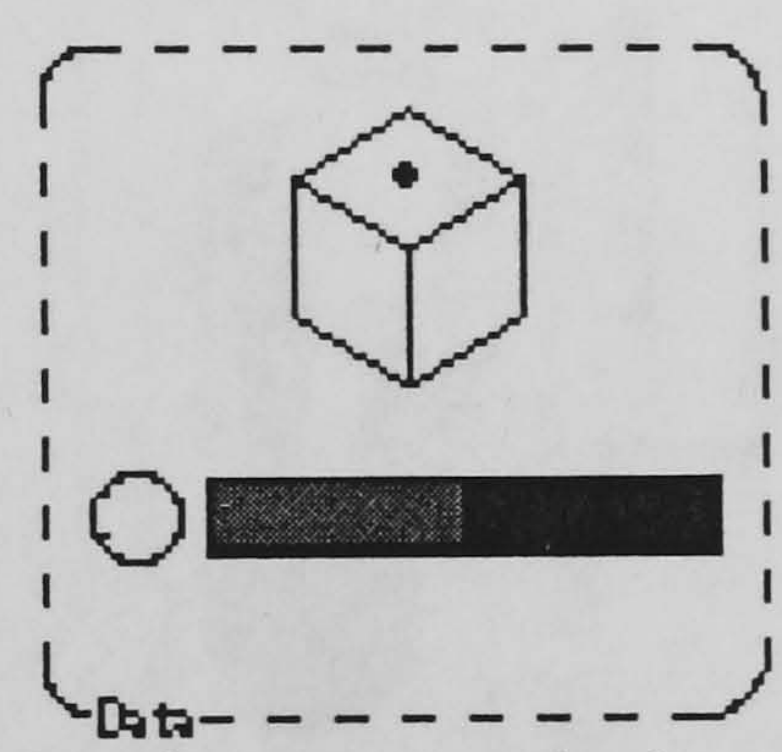

Fig. 5.9 : Each gadget now contained a strength control

The dice gadget, for example, was activated using the strength bar, depicted in Figure 5.8 as a solid black bar with a circular switch at one end. We can imagine the child controlling the strength by allowing a tube (the black bar) to fill with a red fluid until the switch is clicked (Figure 5.9). The strength of the throw, 50% in this case, is represented by the amount of red fluid.

(iv) Each gadget could be flipped to reveal a common set of tools (Figure 5.10 depicts the example of the coin gadget).

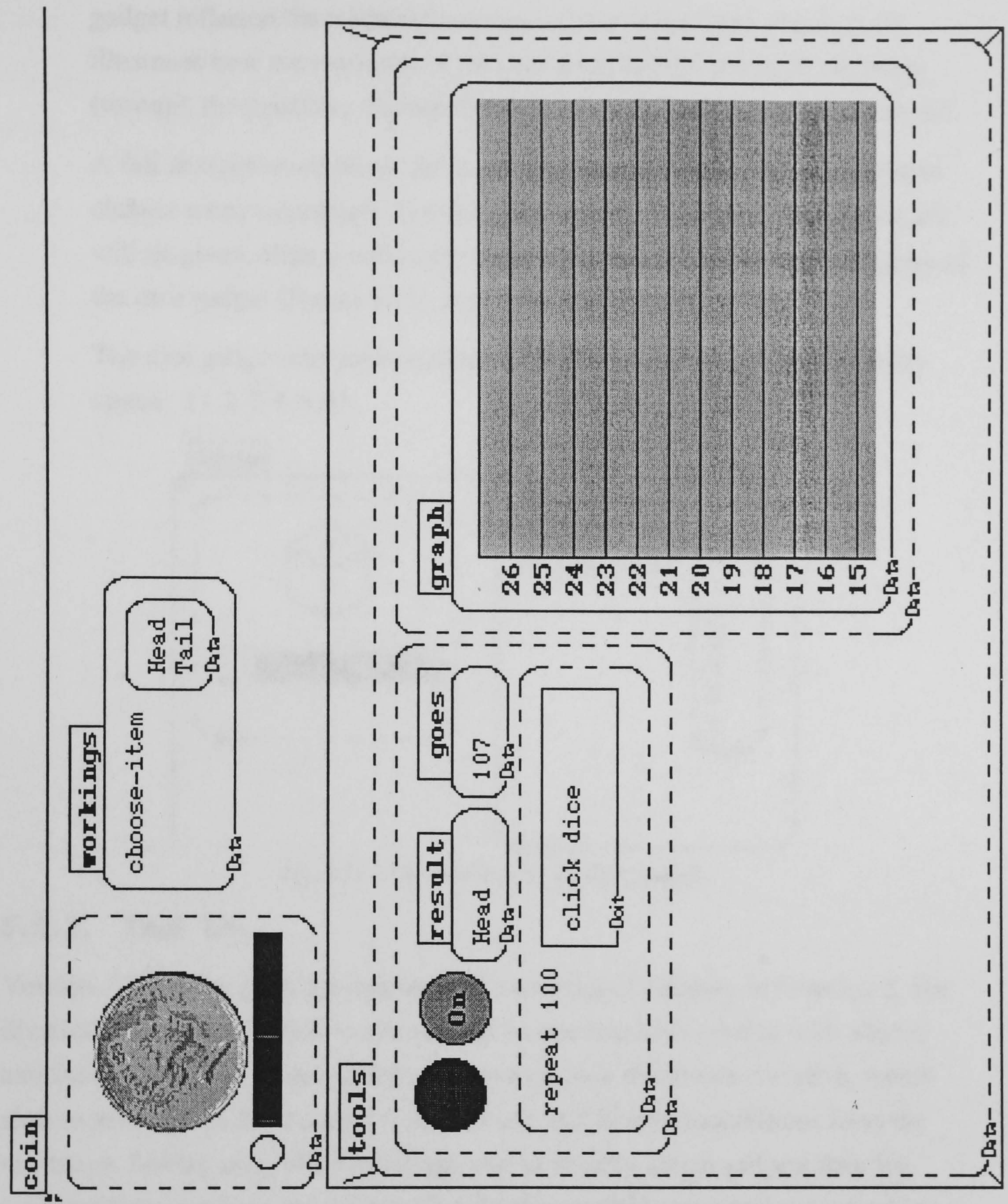

Fig. 5.10 : Each gadget can be flipped to reveal a common set of tools

These tools were contained in a port box so that they could be accessed by any gadget. The graphing tool had now dropped any connection with the roll-a-penny gadget.

(v) The flip-side also contained a copy of the front-side of the gadget. Thus, the children were encouraged to play, whenever appropriate, as if at top-level, with continued access to the strength control.

(vi) The flip-side of each gadget contained a *workings* box. The workings of a gadget reflected the mathematical core of how it operated. Figure 5.10 illustrates how the workings of the coin involved the computer choosing (through the primitive **choose-item**) between the outcomes, head and tail.

A full description of the workings of each gadget will be left until the next chapter when a complete description of the final version of the microworld will be given. Here, I will restrict myself to a description of the workings of the dice gadget (Figure 5.11), referred to in the next section.

The dice gadget was set by default to choose an item from the possibility space [1 2 3 4 5 6].

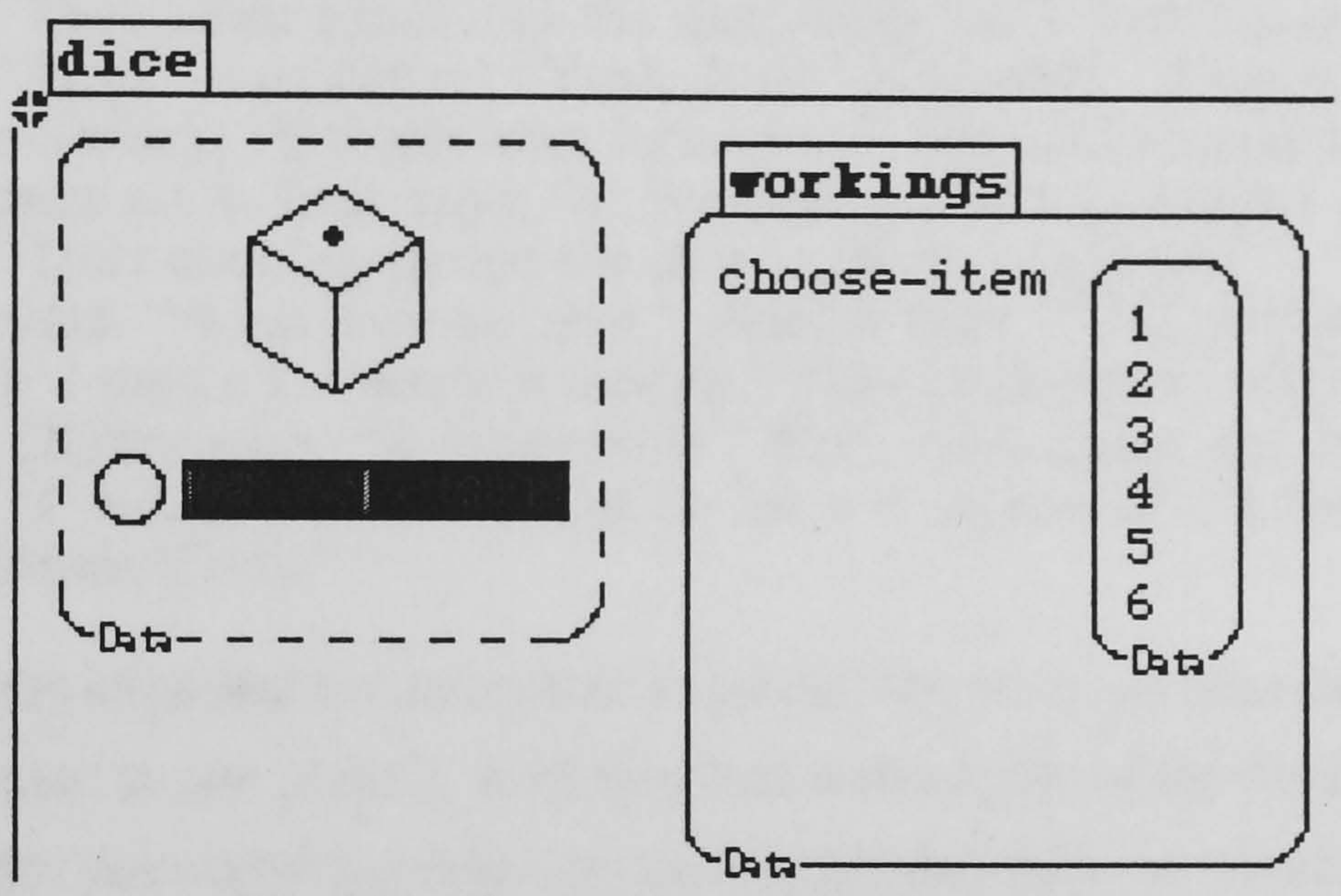

Fig. 5.11 : The workings of the dice gadget

5.5.2. Tool Use

Version 4.20 of the software was used by four pairs of children in Iteration 2. For illustrative purposes, I wish to concentrate on one two hour session with Shirley and Julie, both age 11. The quotations below include discursive narrative, which aims to give a plain and uninterpreted account, and direct transcriptions from the videotape. Shirley and Julie's challenge was to identify which gadgets they felt were working properly and to mend those gadgets which appeared to be working incorrectly. The episode below began with the two children choosing to explore the dice gadget.

The incident as a whole shows how Shirley's and Julie's meanings for *fairness* are closely linked with those of *randomness.* Meanings for fairness were initially focused on actual outcomes but new meanings, related to the theoretical distribution as encapsulated by the workings box, began to emerge. I shall present and analyse the episode in four slices[24], each illustrating a salient issue supported by excerpts from the children's discussions.

Slice 1. "You can't tell what comes next"

The first slice is taken from the children's work as they first began to explore the dice gadget. Shirley and Julie were trying to make sense of how the dice behaved. They were activating the dice using various strengths and sometimes repeating the last strength to gain a feel for the outcomes. At this stage, the children were working at top level, not yet having been introduced to the various tools such as the workings box.

> Julie suggests, "Shall we try it a few times?" They generate a 3 and then a 6, followed by another 6. Julie comments, "Is it Six again? First it was 3 and now it was..." Shirley interjects, "It might be 3 now." They click again and the dice lands on 3. Simultaneously, Julie: "It is!" and Shirley: "Yeah, It is!" Julie adds, "3 again." and says to Shirley: "It might have heard you!" She clicks again and the dice lands on 4. Julie says, "4. Something different almost all the time." They click again and the dice lands on 1. Shirley: "1." Julie comments, "A bit strange, that." Shirley adds, "It'll probably be another 1 then a 4. I doubt it though." They click and the dice lands on 1. Shirley says, "It is actually." They click again and Shirley says, "I wonder if it's going to be a 4 again. It's a bit of a coincidence if it is."

Shirley and Julie expected not to be able to predict the next outcome (although they had some success at one point!). And they had some sense of the distribution, however murky. An important criterion for Shirley and Julie of "working correctly" was that the outcome could not be regularly predicted. I take this as evidence that one meaning for randomness held by Shirley and Julie was unpredictability.

Slice 2: "They should all be the same really"

Slice 2 took place shortly after slice 1 and began by an observation that no 5's nor 3's had been generated and that sixes were quite frequent. At this point, the girls' attention was drawn to the *workings* box, which had until then been ignored. The children noted that there were indeed more sixes in the workings box and continued with **repeat 50 [click dice]** in order to view the result of fifty dice throws. The graphing tool brought the predominance of sixes into focus.

Shirley points at the graph *(Figure 5.12)*, "Woa. There are more 6's .... Yeah, there definitely are more 6's. I don't think the dice is working properly." Julie agrees, "Yeah. It should be around the same for each of them. The 5's and the 3's are the same but ...." I ask, "Right. Right. And what about the others? Do you feel they're OK — the 1, 2, 3, 4, 5?" Shirley responds, "Don't think they are. I think they should all be the same really." Julie agrees, "Yeah. I think 3's, 1's and 5's are OK." Shirley: "Yeah." Julie continues, "...and the 2's are sort of OK., yeah, but it's the 4's and the 6's that are a little bit over the top." Shirley adds, "Well, the 4's — the 6's have got too many and there's ones that haven't got enough."

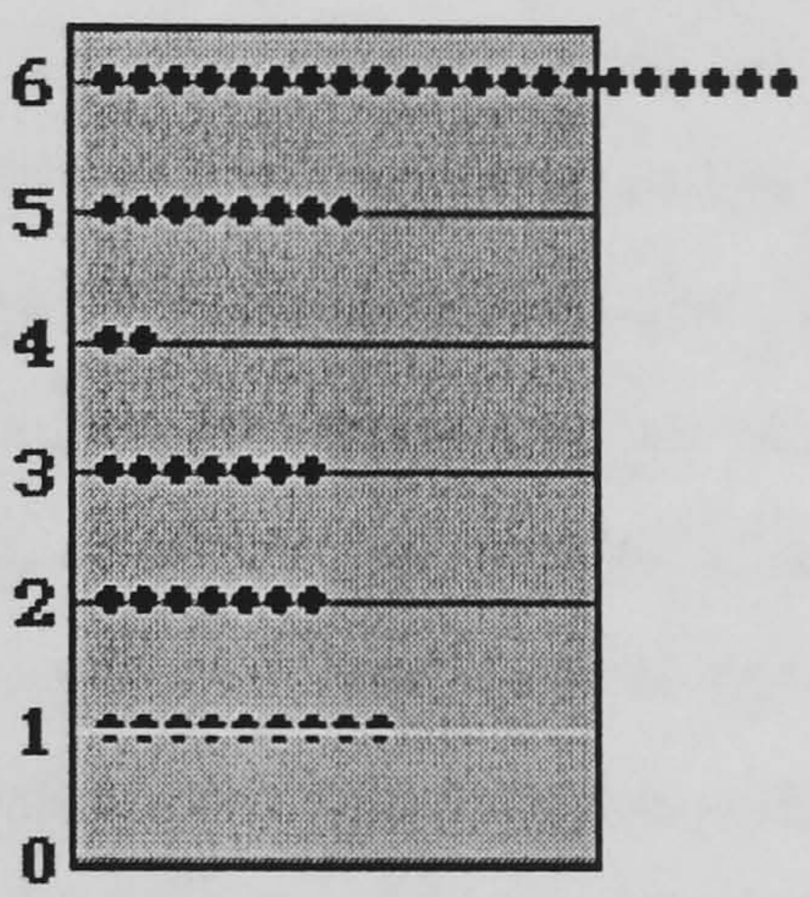

Fig. 5.12 : "Woa. There are more Sixes"

The graph confirmed for them, even though the evidence was quite limited, that the dice was not working properly. However, they not only saw problems in the number of sixes but also in the frequencies of some other scores. Shirley and Julie decided at this point to set about mending the dice gadget. They edited the workings box and, after 50 trials, generated another graph (Figure 5.13).

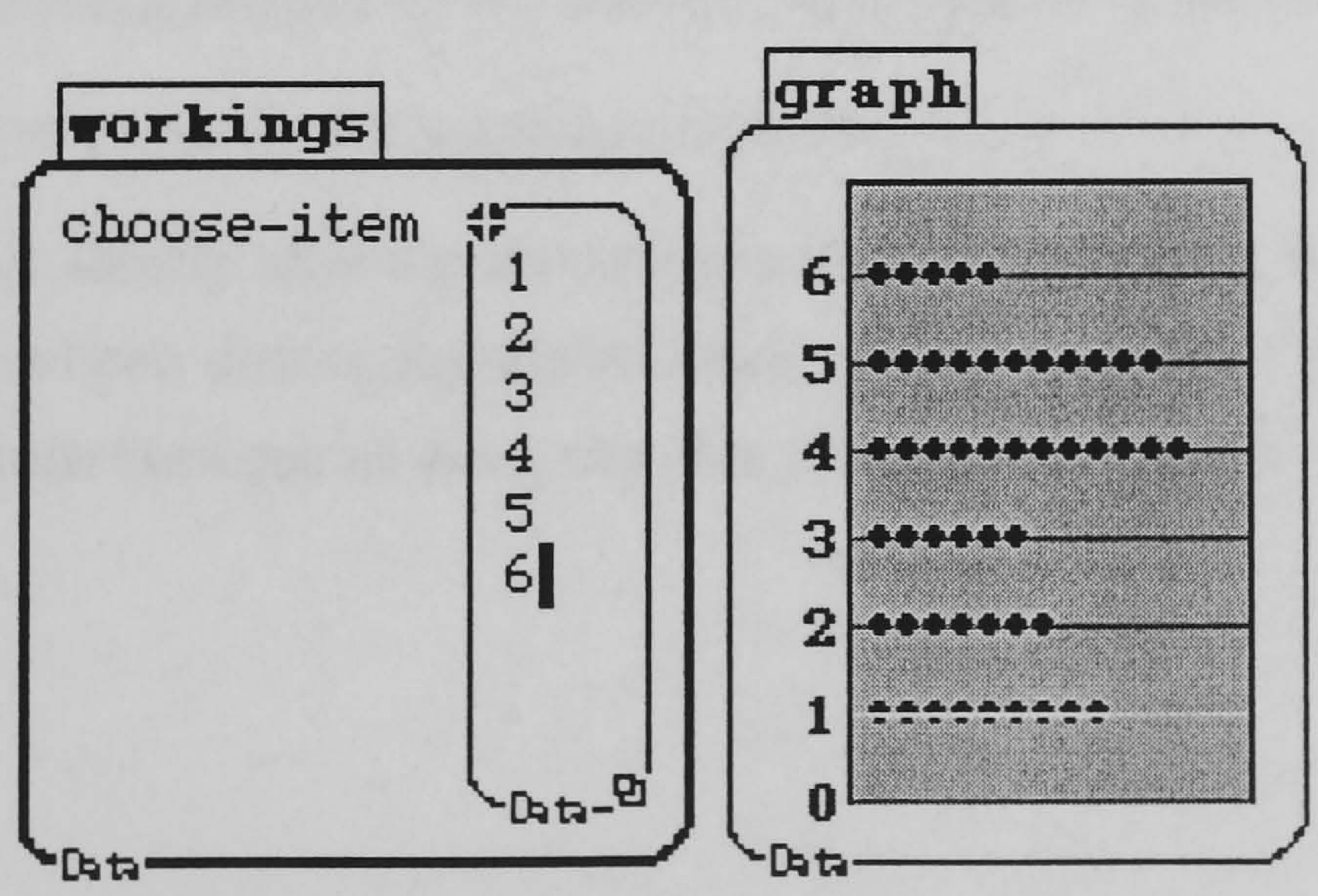

Fig. 5.13 : After 50 trials, they generated another graph

From our 'expert' point of view, this situation represents a fair dice because we focus on the workings. However, Shirley and Julie's reaction to the graph indicated how differently they understood the situation.

> They point to the rows of 5's and 4's. Julie comments, "It's a bit better there." Shirley says, "But we haven't got enough 6's now." Julie agrees. Shirley adds, "And also there's too many 4's." Julie suggests, "If we added one more 6 on to the em ..." Shirley: "Yeah. But still, 5's and 4's are too many, though, aren't they?" Julie agrees, "Yeah. They need to be a bit less. But you can't make them less, can you!" Shirley interjects, "No." Julie continues, "... cos then there'd be nothing. Shirley says, "Unless you take one away — no, because there isn't any more to take away." They return to the workings box and add another 6.

Their focus was on the results, which were more salient for them than the workings as a means for understanding the distribution of the outcomes. The girls' intuition was to correct for the discrepancy between the number of times different scores had occurred by adjusting the workings accordingly. The girls then iterated, three times in all, through a process in which they amended the workings box, repeated fifty clicks, looked at the graph and changed the workings box accordingly. Shirley's and Julie's initial meanings suggested to them that fairness should result in equal, or very nearly equal, frequencies for each possible outcome. So the workings were used as an *input* tool; a change in the workings caused a change in the graph but it was the graph which validated the fairness or otherwise of the dice.

The first two slices through this episode illustrate how the expressive domain enabled the children to hold their meanings open to scrutiny through the window of the workings box, how the children's intuitions shaped their tool use. However, the next slice illustrates the complementary process: how the children's meanings were themselves being shaped by the structuring resources in the domain.

Slice 3: "It's not necessarily not working properly"

Slice 3 took place shortly after the above three attempts to modify the workings. The girls had just been discussing the meaning of **choose-item**. I decided to make explicit connections between an everyday dice and the dice gadget.

> I ask, "OK. If you threw a real dice, fifty times — OK.— like you're doing — What do you think might happen? If it was an ordinary, fair dice, what do think would happen, if you did it that number of times?" Shirley suggests, "Well, I think — I don't even know — I don't think it would turn out so that it's all the same because you can't say, well — if it's fair — that it's gonna turn out all the same. Because if it's fair, then it's gonna be random, so ..." Julie offers, "I think it would be like umm, quite fair, but it wouldn't be exact." Shirley says, "You wouldn't be able to say well, it's gonna be so many 6's — there's gonna be eight 6's, there's gonna be eight 5's, eight 4's, eight 3's." Julie adds, "You might have like, ten of ..." I ask, "Right. So in your workings at the moment that you've got, do you think that's a fair situation that you've — the way you've written the workings? You've written '**choose-item [1 2 2 3 3 4 5 6 6]**.' They both say, "No." Julie adds, "Maybe we should do like two of everything." Shirley: "Or one of everything. Because otherwise it's not gonna be fair, is it?"

Shirley and Julie modified the workings box (Figure 5.14) in accordance with Julie's suggestion.

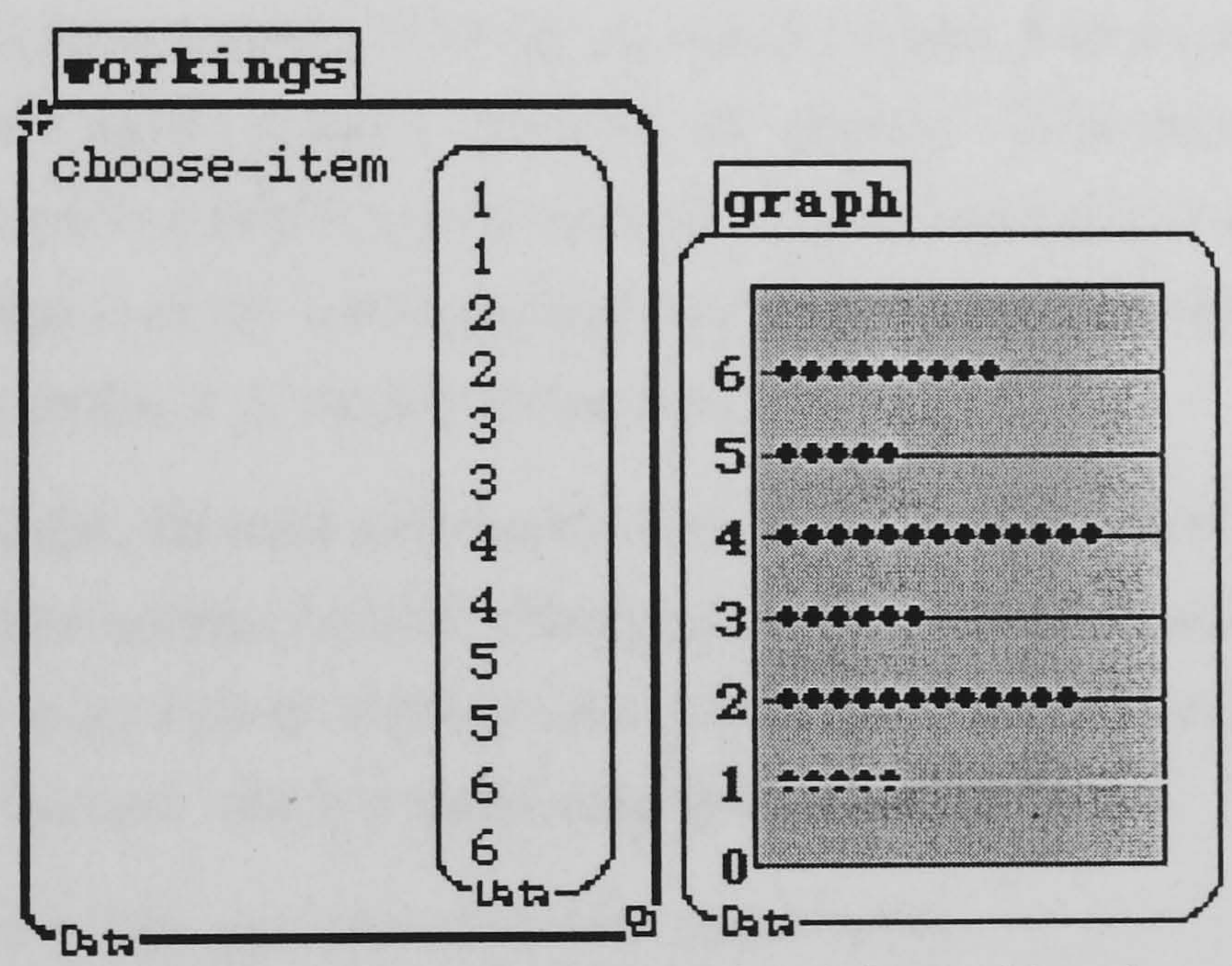

Fig. 5.14 : Shirley and Julie modified the workings box

However, the above discussion had emphasised a connection between the workings box and fairness, in addition to the connection between the graph and fairness. An immediate reaction was again to be concerned that there were too many 4's and 2's. However, such concerns were cut short by Shirley's pivotal insight.

> "I've just thought of something. It's not necessarily not working properly, because it's got to be random, hasn't it?"

Shirley recognised that fairness in the workings box did not necessarily imply fairness in the data. Random behaviour might result in variation between the frequencies of different results even when the dice was fair. Shirley's assertion needed some checking as we see in the next slice.

Slice 4: "No patterns in outcomes"

Slice 4 took place towards the end of the episode with the dice gadget. By this stage, and after much experimentation, Shirley and Julie had mended the dice gadget by editing the workings to read: **choose-item [1 2 3 4 5 6]**. They were now checking whether the dice really was working properly. They referred to the results box, which simply contained the history of their interactions with the dice gadget in the form of a list of outcomes.

> Julie suggests, "We could have a look at this." She points to the list of numbers in the *results* box. The fifty results were listed in a box called *results*. Shirley and Julie scrolled through the list. Julie comments, "It looks about normal." I ask, "What do you mean when you say that?" Julie says, "It looks quite random — because *they're not* all in groups or anything." Shirley comments, "In quite a bit there's like so many 2's together or so many 4's or whatever, but it definitely looks random. It doesn't look as if they've got all the 1's at the top or the 2's or the 3's ....." I clarify, "Right. There's no obvious pattern to it." Julie: "No."

What Shirley and Julie seemed to be saying was that there was no obvious pattern in the results, and that this made it "random" or "normal". This assertion (that a lack of pattern is an aspect of random behaviour) is seen as consistent with their earlier assertion that fairness in the workings box does not necessarily imply fairness in the data, another attribute of random behaviour.

For Shirley and Julie, fairness and randomness were closely related, although, as we have seen, these notions became disaggregated as they worked with the microworld. We were able to observe three aspects of Shirley's and Julie's intuitions about fairness which underpinned their thinking:

- if the dice is fair, you can't tell what comes next;
- if the dice is fair, there are no obvious patterns in the sequence of results;
- if the dice is fair, the different outcomes should be equally, or nearly equally, represented in the results.

An extra meaning for fairness seemed to emerge during the interaction:

- if the dice is fair, the different outcomes should be equally, or nearly equally represented in the workings box.

5.5.3. Emerging issues

The specificities of the environment were crucial in supporting the forging of mathematical meanings in use, the webbing process. Throughout the tools use phase of Iteration 2, the resources of the microworld which proved crucial were:

- the **repeat** primitive, which supported a move towards aggregation of results;
- the *graphing tool* which supported the visualisation of the results in terms of variations in aggregated frequencies (e.g. there were too many sixes);
- **choose-item**, the primitive designed to empower explicit articulation of the notion of random choice among a set of numbers;
- the *workings box*, which provided an interactive representation of the distribution (or, at least, a way of thinking about the distribution).

Meanings associated with control and predictability, seen in Iteration 1, were once more apparent in Iteration 2. In addition, fairness and the irregularity of past results emerged as further ways of discriminating between randomness and non-randomness.

5.6. ITERATION 3 : ANALYTICAL

5.6.1. Tool Development

In Iteration 2, there was evidence that structures, such as the **repeat** and **choose-item** primitives, and tools, such as the graphing facility and the workings box, shaped children's meanings for the behaviour of stochastic phenomena. Uncontrollability seemed to take on enhanced dimensions, as did fairness. There was however little evidence of the reshaping of unpredictability. There is a sense in which random behaviour over the long term is characterised by increased predictability — the relative frequency of an outcome in the long term resembles, with ever increasing reliability, the proportion of that outcome in the possibility space. I conjectured that children would be more likely to co-ordinate their unpredictability meaning if enhancements were made to the graphing tool. Development of the graphing tool is the first of several important modifications to the tools, set out below.

(i) I had been struck by how some children in Iteration 2 had been drawn to the differences in the lengths of rows in the pictogram display rather than the global similarity of the rows (see, for example, Shirley and Julie's discussion of the graph in Figure 5.13). The pictogram accentuated unpredictability rather than pointing towards a sense of predictability in the longer term[25]. After considering various options for enhancing the graphing facility, I decided that a pie chart would better display the proportional nature of predictability in long term behaviour; that is to say that the size of the sectors in the pie chart would stabilise in the longer term in just the same way as the relative frequencies increasingly reflect the proportional representation of outcomes in the workings box. In hindsight, this now seemed such an obvious tool to provide in a stochastic domain.

(ii) The independent exploration of each gadget was now a central function of the Chance-Maker microworld, and it seemed unnecessarily restricting to have a single graph and results box, common to all the gadgets. It now seemed a neater solution for the autonomy of the gadgets to be reflected in separate tools for each gadget. Thus, in Iteration 3, each gadget was given its own graphing facility, results box and so on. One benefit would be that children could return to a gadget that they had been using previously and check what results they had obtained, perhaps to compare them with recent results generated by the latest gadget.

(iii) The two-dice gadget had originally been created because of its symmetric but non-uniform probability distribution, which might be seen as having similar characteristics to the distribution of distances rolled by the coin in roll-a-penny or travelled by the Frisbee. Interviews had suggested that many children saw the various totals for two dice as equiprobable. (The interviews from Iterations 2 are not reported but transcripts of the interviews for Iteration 3 are included in Appendix 3.2 or on the world wide web. An example of how these interviews were interpreted is given in section 9.2.1, where Anne and Rebecca are seen to be one of several pairs who describe the various totals for two dice as equally easy and equally hard to obtain.) Unfortunately the children in Iteration 2 had scarcely engaged with the two-dice gadget. One reason was that I had allowed the children a great deal of freedom in the clinical interviews to work with gadgets of their choice. They had opted away from the two-dice gadget largely, I suspected, because the workings looked complex.

Because of the creation of the two-dice gadget, I had become keen to explore in Iteration 3 how children would make sense of the totals but I was concerned that the multiplicity of outcomes for two dice would hinder sense-making. I decided that I needed a new gadget, which was a simplified version of the two-dice gadget, to act as preliminary support. I created a two-spinners gadget; each spinner was numbered 1, 2, and 3. The two-spinners gadget reported the total of the two spinners.

(iv) Given the nature of task — to identify and mend broken gadgets — it occurred to me that children would be attracted to engage with gadgets which worked 'improperly' when they were first encountered. I therefore made changes to the default setting of the workings box in several gadgets. For example, the dice gadget now began with a bias towards 6's. The default settings are all detailed in the next chapter.

(v) The spinner gadget seemed to offer more initial support than the dice gadget, no doubt because it's appearance was more iconic in the sense that the sectors of the pie chart gave an immediate visual image of fairness or lack of fairness. The spinner gadget was re-positioned before the dice gadget in the top-level appearance of the box of gadgets, and I resolved to specify that the children worked with the spinner gadget before using the dice gadget.

These changes were embedded in Version 5.30 (Figure 6.1), which is described in detail in the next chapter, and was the version used during the tool use phase of Iteration 3, analysed in Chapters Seven to Nine. Version 5.30 and latest version of the microworld, Version 5.33, containing minor changes from that used by the children, can be obtained from the World Wide Web at address: http://www.warwick.ac.uk/wie/staff/DP.htm

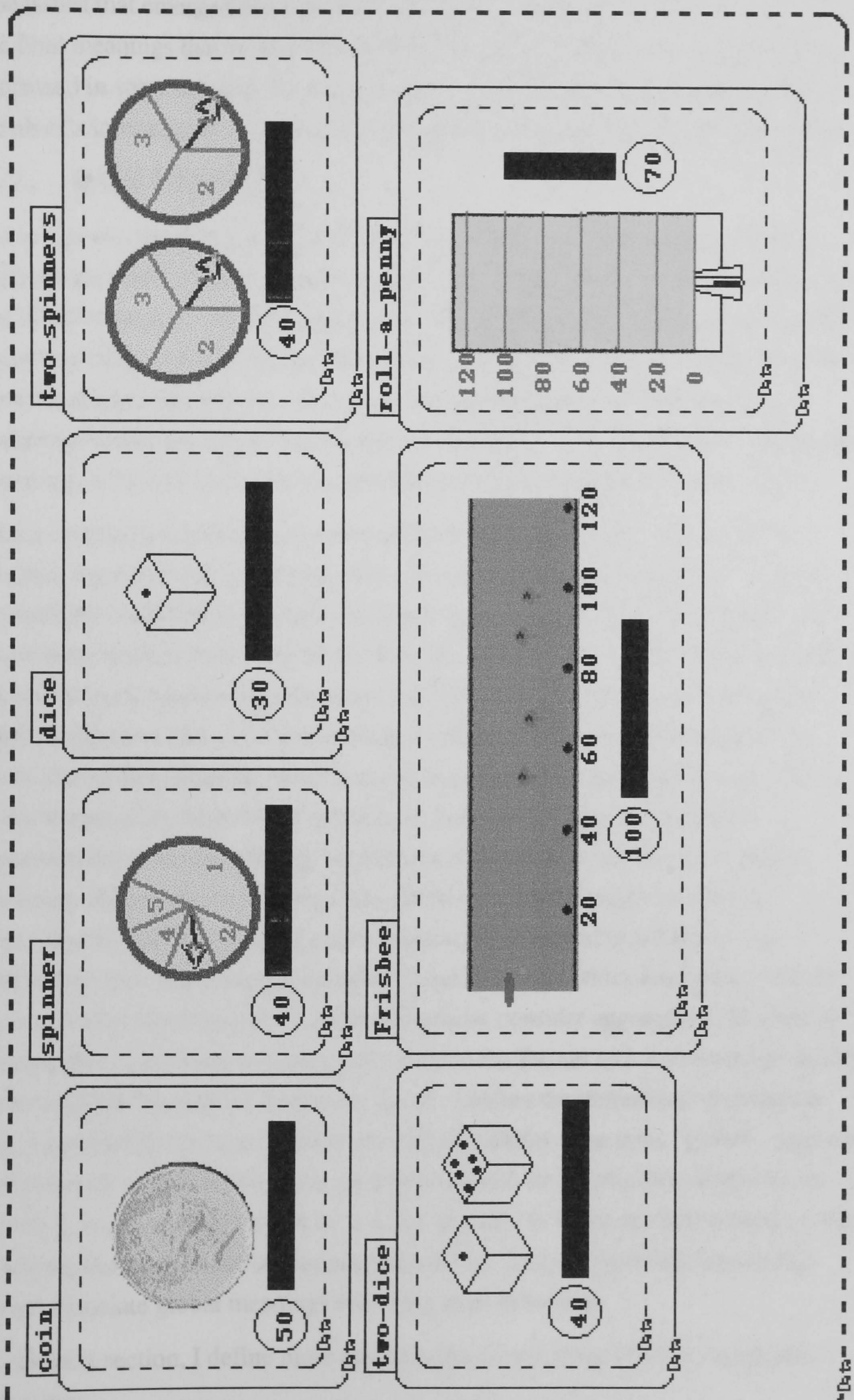

Fig. 5.15 : The tools in Version 5.30

The issues that emerged from the tool development phase for Iteration 3 represent the final meanings that were constructed *for* the Chance-Maker microworld and are discussed in some detail in the next chapter. The meanings that emerged from the children's interactions *in* the microworld are described in Chapters Seven to Nine.

5.7. DISCUSSION

An image was emerging, in which webbing with the tools and structures of the microworld enabled the construction of new meanings, which were connected with the initial meanings, for randomness, but which lent heavily for expression on the resources made available in the microworld. It was as yet unclear exactly how the new meanings related to the initial meanings and just how situated these new meanings would be. These were questions for the next iteration when the children's meanings would be studied more systematically (see Chapters Seven to Nine).

The increasing prominence of meanings for random behaviour, such as lack of control, unpredictability, fairness and irregularity of past results seemed to relate primarily to variation from trial to trial, perhaps abstracted from the experience of short-term random behaviour. There was some evidence that these meanings took on new dimensions during interaction with the computer-based resources. One might conjecture that these new meanings might, with further refinement of the tools and more systematic observation of the children's actions, be seen to touch upon longer term behaviour. Certainly, in Iteration 2, Shirley and Julie's distributional-oriented meaning for fairness, in contrast to the outcome-related meaning, might point in this direction. Such a conjecture might be seen as supported by Gill's and Jenny's new meanings for control in Iteration 1. A distinction then was emerging between those meanings which focused on trial by trial variation and those which might potentially consider aggregation. In order to discuss these two types of meanings, I refer to the former as *local meanings* and the latter as *global meanings.* The term, 'local', stresses the immediacy in meanings like unpredictability, uncontrollability and irregularity. The term, 'global', captures the potential to look at the trends or proportions in the aggregation of results. In practice, local meanings might be used by children to make sense of both short term and long term behaviour of random phenomena, though expert-like knowledge might associate global meanings with long term behaviour.

In the next section, I define more precisely the terminology of local and global meanings.

5.7.1. Local Meanings

Stochastic phenomena exhibit certain behaviours in the short term. The meanings that we can construct from this short term behaviour concentrate on the here and now, local in the sense that such meanings focus on trial by trial variation. Local meanings have a number of characteristics:

- The next outcome is not predictable, though some apparent success in the short term may be experienced *(unpredictability)*,
- No patterned sequence in prior results is sustainable over the longer term *(irregularity)*.

These two meanings are closely related. Though the former looks to the future whilst the latter considers historical data, predictions are often based on previous experience. A third characteristic of local meanings is tied into the observer's own influence over events:

- The observer is unable to exert physical control over the outcome of the phenomenon *(unsteerability)*.

By physical control, I refer to factors which give you a direct feeling of control, like how you throw the dice, or toss the coin. To highlight the personal and tangible dimensions of this type of control, and to distinguish it from its counterpart in global meanings below, I refer to it as *steerability* (or in the case of random phenomena, *unsteerability*). On the computer, this sort of control can only be realised through a device such as a mouse or a joystick. The main point here is that, if I believe that I can direct the outcome through my own physical actions, then I am unlikely to regard the phenomenon as stochastic.

5.7.2. Global Meanings

Global meanings focus on an aggregated overall view of the stochastic. Global meanings are characterised by the following properties:

- The proportion of outcomes for each possibility is predictable *(probability)*,
- The proportion of prior results for each possibility in the possibility space will stabilise as an increasing number of results is considered *(large numbers)*,
- The observer is able to exert control over these proportions through manipulation of the possibility space *(distribution)*.

There is an interesting correspondence between each set of meanings. The most striking aspect of this correspondence is that each local meaning is inverted in relation to its global counterpart. Thus, unpredictability as a local meaning is inverted in comparison to the global meaning of predictability (in a proportional sense). Similarly, control can not be exerted locally, whereas there is a global meaning for control through manipulation of the distribution.

5.8. SUMMARY OF CHAPTER FIVE

This chapter has set out the evolution of a piece of software through three iterations. (The final stage of this evolution will be detailed in the next chapter.) In parallel, my understanding of the children's meanings for the stochastic has deepened. The story of Chapter Five is one in which these two strands shaped and were shaped by each other.

The microworld evolved through three main stages:

(i) The first stage was Iteration 0 in which some initial tools were based on previous work associated with active graphing. The software at this stage was marked by an emphasis on game design, which offered a strong sense of purpose to the children but was found not to engage the children in interacting with the concept of distribution, and so was weak in terms of supporting children's construction of utility for distribution.

(ii) Iteration 1 was marked by rapid and successive re-conceptualisations of the software. The introduction of direct programming control of the distribution was seen as too complex. Instead, new tools were brought into the environment to enable the expression of how roll-a-penny behaved in terms of the behaviour of other phenomena. The range of actions seemed to be too complicated and so I sought simplification. The **click** primitive, as an action common to all tools, provided the required simplification.

(iii) The third stage of tool development saw the invention of a collection of devices, called gadgets, which could operate autonomously and could be controlled directly through a strength control or through manipulation of the workings box. These gadgets, through physical appearance and dynamic actions, represented close approximations to their everyday counterparts.

By using these tools, children articulated meanings for randomness which included unpredictability, unsteerability, irregularity in previous results and fairness. During interaction with the resources in the microworld, the meanings seemed to take on new characteristics. Meanings of control appeared to be enhanced by the possibility of control through the workings box. Meanings of fairness began to manifest themselves initially through roughly equally long rows of results in the graphing tool but then also through the appearance of the workings box.

There was evidence that the **repeat** and **choose-item** primitives, alongside interaction with the graphing tool and the workings box, were centrally important in the process by which these new meanings emerged.

Whilst some co-ordination of meanings of control and fairness had been observed, there was as yet little movement towards an analogous co-ordination in meanings of unpredictability. New meanings for unpredictability could perhaps emerge through the longer term behaviour of the gadgets, which draws attention to the fact that the initial meanings of unsteerability, unpredictability and irregularity relate epistemologically to the short term behaviour of stochastic phenomena.

The fraction of the children's work reported in Iteration 0 to 2 was intended to illustrate a growing awareness of a deep relationship between the resources provided and the co-ordination of meanings for randomness. It was clear from the tool use as a whole that further improvements to the software could be made, though it was not envisaged that future changes would involve major revision, and these will be discussed in the next chapter. On the contrary, the move towards the notion of a bundle of gadgets, each of which could be investigated in its own right, seemed to offer exactly the right environment to study not only the meanings for the stochastic in a specific situation but also how those meanings were articulated from setting to setting, from gadget to gadget.

In Iteration 3, children's meanings for the stochastic and their relationship to the specificities of the microworld were studied more systematically. Chapter Six sets out how the tool development phase responded to the software design issues emerging towards the end of Iteration 2 and sets out in detail the final version of the software, before the tool use phase of Iteration 3 is discussed in subsequent chapters. During Iteration 3, I named the software, the *Chance-Maker* microworld, to emphasise the way in which the children in the later iterations became increasingly engaged in controlling (as it were 'making'), rather than merely reacting to, chance.

NOTES

21 A doit box is a Boxer term for a box which includes an executable procedure.

22 The angles for the wheel are presented as a special type of box, called a port. In Boxer, a port gives a view of a box sited elsewhere in the software. The port is 'hot wired' to the original box so that changes in either the port or the original box cause immediate and corresponding changes in the other.

23 This incident has previously been reported elsewhere (Pratt & Noss, 1996)

24 This incident has also previously been reported elsewhere (Pratt & Noss, Under review)

25 Indeed, as the number of trials increases, the variation between the frequencies of the possible outcomes will become greater in absolute terms. The pictogram will reinforce increases in variation of this sort. The challenge was seen as finding a representation which pointed towards the decreased variation in the relative frequencies of the different possibe outcomes.

CHAPTER SIX

Meanings *For* A Domain of Stochastic Abstraction

6.1. OVERVIEW

In this chapter, I describe in some detail the final version of the Chance-Maker microworld, which was used by children in Iteration 3. The approach is to discuss key principles which, as I look back over the iterative design process, have underlain that development, becoming embedded in the final version of the software. These principles can be thought of as meanings that have been constructed *for* a domain of stochastic abstraction.

Where these principles are most easily explained through examples of what children did, I refer back to the work discussed in Chapter Five or forward to examples of work discussed more fully in Chapters Seven to Nine.

I will now discuss the meanings for the domain of stochastic abstraction, as constructed through the iterative design process. In so doing, I will simultaneously be able to define the Chance-Maker microworld, a description which will inform reading of the subsequent chapters.

6.2. MEANINGS FOR THE CHANCE-MAKER MICROWORLD

As I reflect on the iterative design process from Iteration 0 through to Iteration 3, it is possible to identify epistemological and pedagogic forces driving the design process forwards. The Chance-Maker microworld can be seen as a convergent solution, even a compromise, between two needs:

(i) to address the mathematical content embedded within the stochastic domain, and

(ii) to engage the child in a process which generated meanings for that content.

In general, these needs can at times seem to be in conflict since the more rigorous the mathematics the less meaningful it might appear. The solution involved the re-shaping of the mathematics into new formalisms, which could be manipulated in meaningful and purposeful ways by young children. In this section, I will first address what the new mathematical formalisms looked like in practice (section 6.2.1). I will then consider how these formalisms were constituted to maximise the likelihood of engagement by children (sections 6.2.2 and 6.2.3), and finally, but crucially, the nature of the tools which empowered those children to construct meanings for the formalisms during use (sections 6.2.4 and onwards).

6.2.1. Designing New Formalisms

An explicit intention within the Chance-Maker microworld was to blur the distinction between the informal and the formal. The challenge was to design an environment in which the child could play informally and yet encounter and make connections with formal expressions of the stochastic. These new formalisms turned out mostly to be unconventional in nature but represent potent ways of expressing and using chance, at least within the confines of the Chance-Maker microworld.

For each gadget, the workings box contains an explicit representation of an underlying probability distribution. At the same time, the workings box formalises the essence of how the gadget works; it contains the main instructions, which tell the gadget how to decide the outcome when it is activated. This dual role for the workings box provides the linking mechanism between the formal and the informal. Let us consider each gadget in turn.

The coin gadget

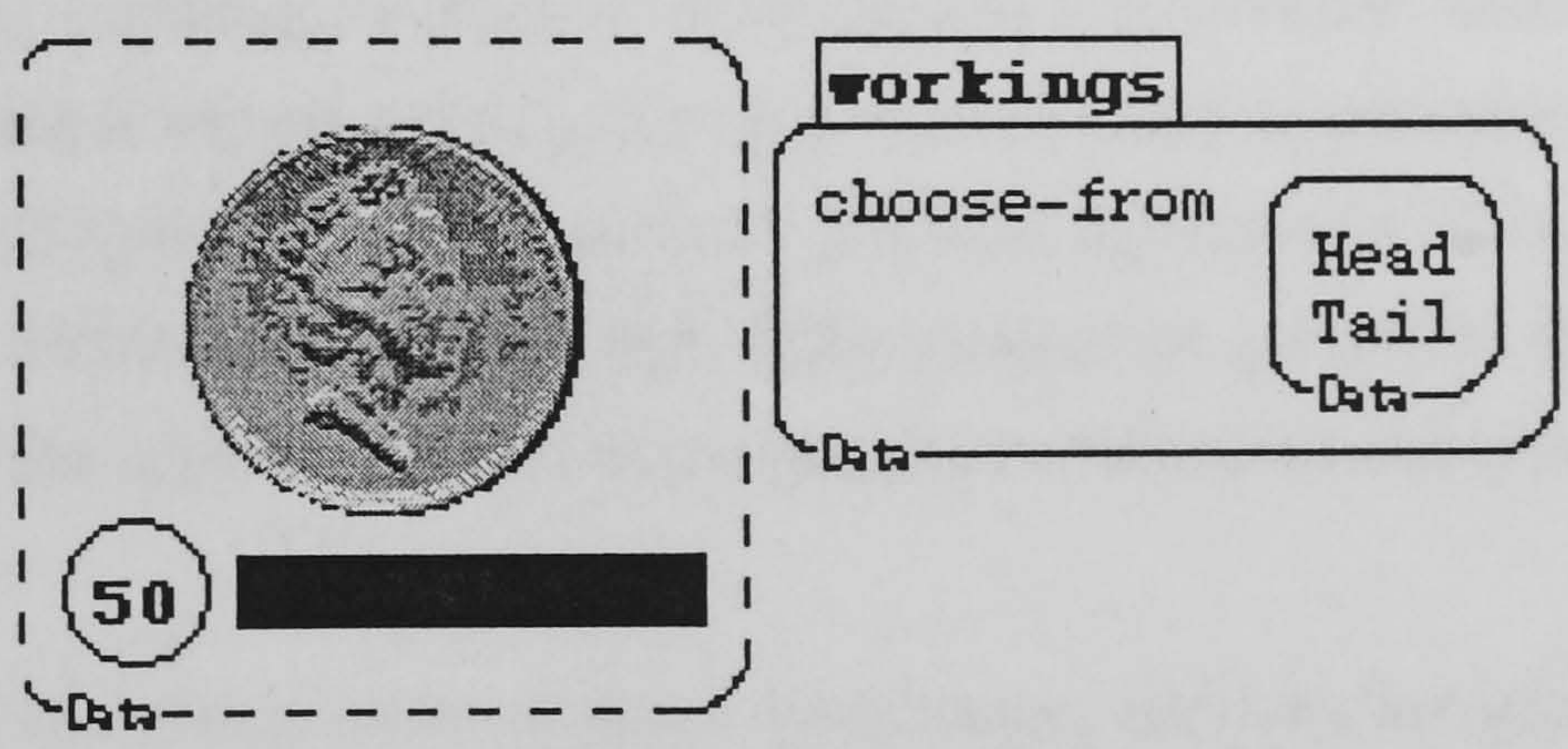

Fig. 6.1 : The workings of the coin gadget

The coin's workings indicate that the coin has to choose between head and tail (Figure 6.1). The fundamental primitive has been altered from **choose-item** to **choose-from** (the word *item* had occasionally been confusing for some children).

For the initiated, the workings seem to say very little: simply choose between heads and tails. Indeed, this is all there is to say. The formal expression that a random choice is made from the items in the data box (with equal probability assigned to each item) contains a full and complete explanation of the process. The primitive **choose-from** contains within itself the essence of randomness. For the uninitiated though, its meaning can only be discriminated through use (see the *designing for* *meaning through use* section for a detailed discussion of this critical issue).

The coin gadget was offered during Iteration 3 as an exemplar to introduce children to the structures and tools within the gadgets in general. It was not intended to be problematic but rather a straightforward case with which children could familiarise themselves with the tools. We will see in later chapters that the coin gadget was not always as straightforward as I had anticipated.

The spinner gadget

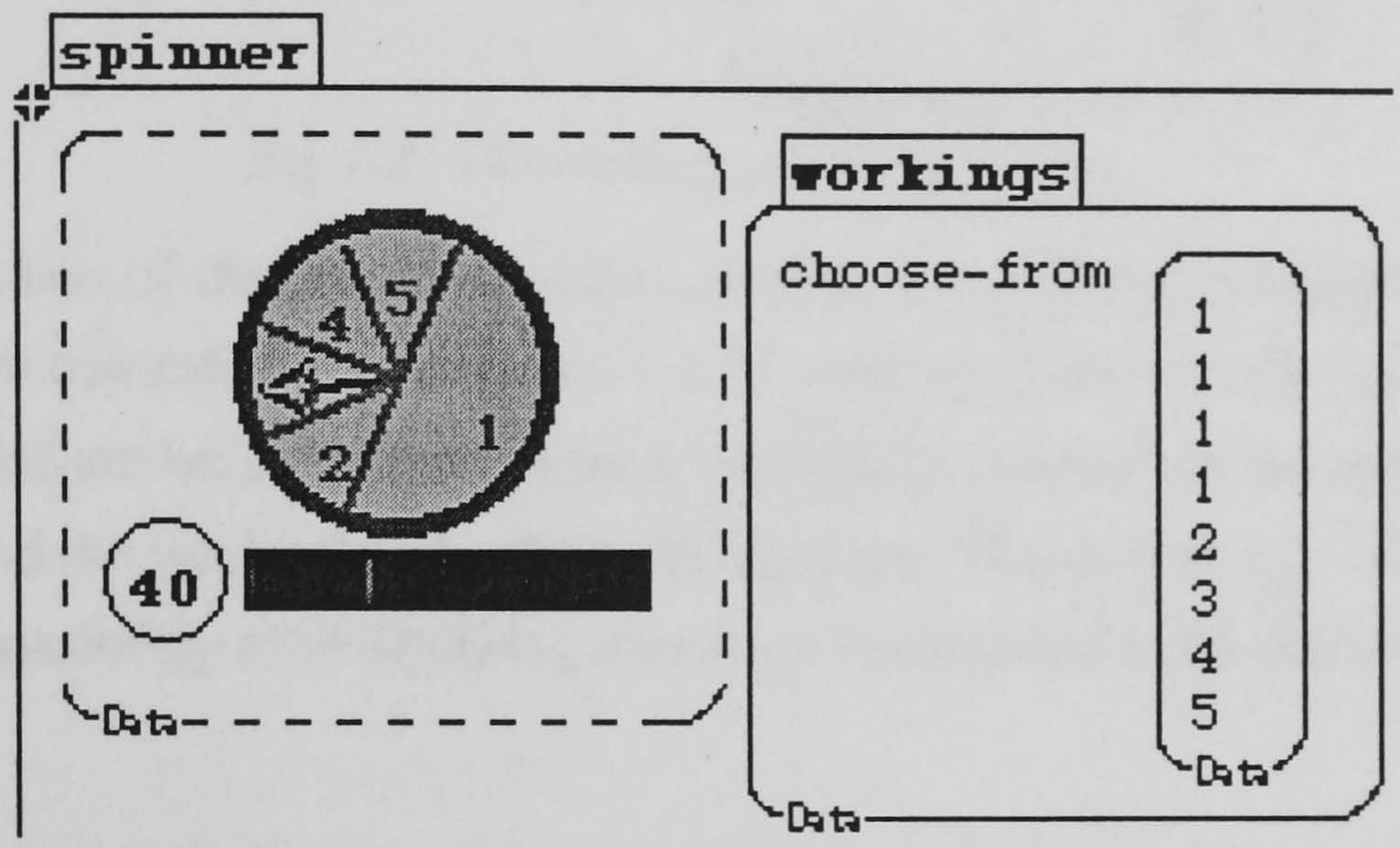

Fig. 6.2 : The workings of the spinner gadget

The spinner gadget, like several others, presents a formalisation of non-uniform distributions by building on notions of uniformity and fairness. Bias towards certain outcomes is represented by the repetition of items as necessary in the workings box (Figure 6.2). The spinner's physical appearance reflects the configuration within the workings box. This connection is reinforced since the appearance of the spinner changes automatically to reflect amendments in the workings box.

By presenting as default a non-uniform distribution, children are introduced to this representation of non-uniformity, whilst the unfair appearance of the spinner may encourage interaction with the workings (see the *predictability* local meaning in Chapter Seven).

The dice gadget

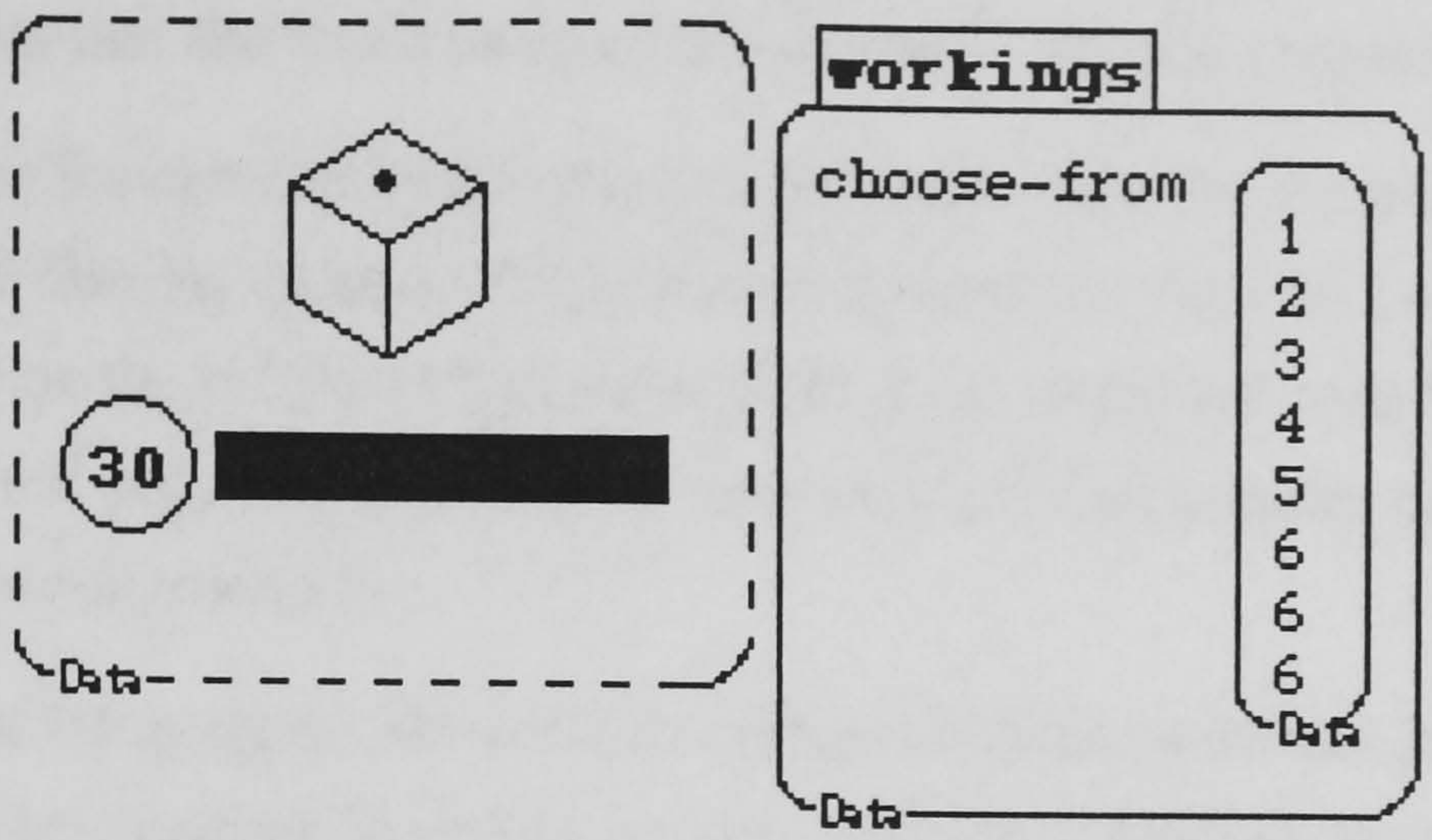

Fig. 6.3 : The workings of the dice gadget

The workings box of the dice represents a similar non-uniform distribution, though the bias is now towards 6's rather than 1's. In contrast though to the spinner, the sides of the dice are hidden and so there is no explicit connection between the appearance and the workings box of the dice gadget. The dice gadget offers children the possibility of re-applying meanings constructed from activity with the spinner.

The two-spinners gadget

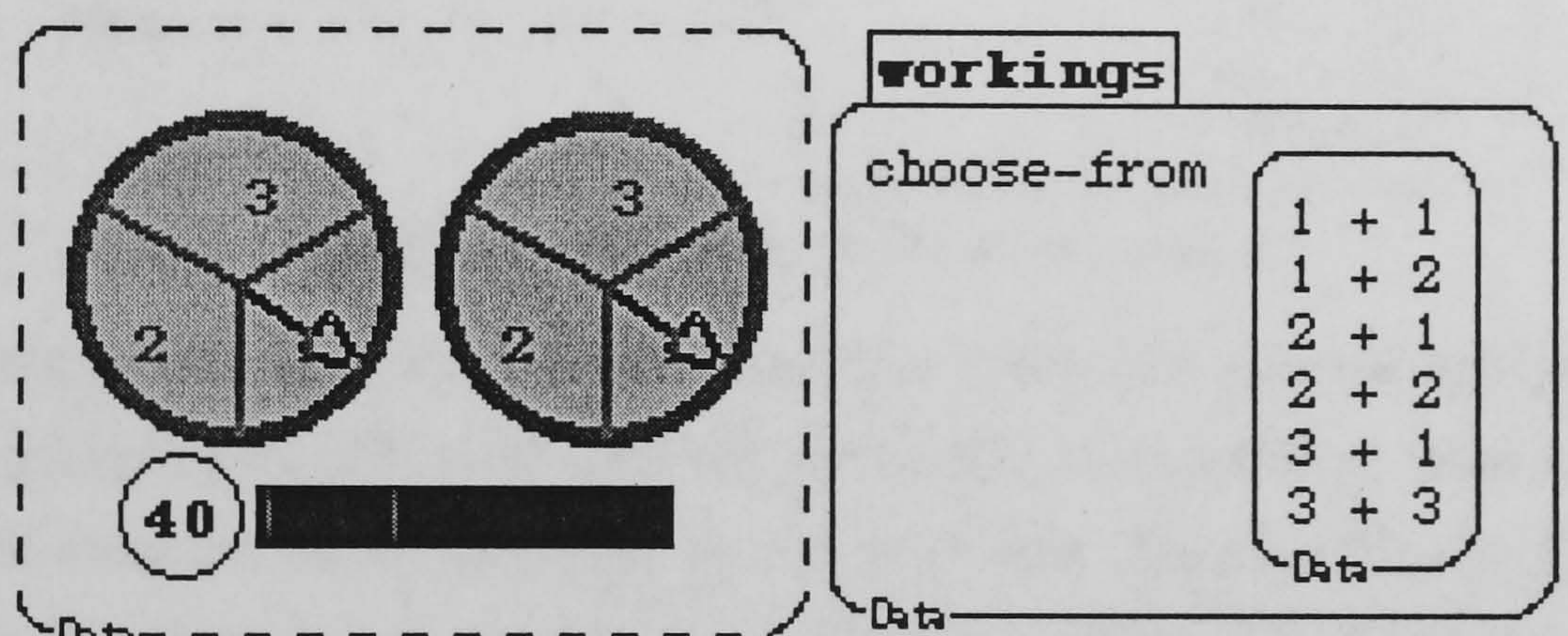

Fig. 6.4 : The workings of the two-spinners gadget

This is a new gadget designed, as a simpler version of the two-dice gadget, to enable the observation of children's evolving meanings for the behaviour of the total of two spinners. The form of the workings of the two-spinners gadget is intended to direct attention towards the combinations, which make up the totals from 2 to 6 (Figure 6.4). Thus, an outcome of 1+1, chosen from the workings, would correspond to a one appearing on each spinner, giving rise to a reported result of 2.

The default setting for the workings includes 2+1 as well as 1+2, to intentionally raise the issue of whether one is redundant. It was anticipated that some children might believe that fairness implies that every possible combination should be included equally often in the workings box, whereas other children might argue that fairness suggests that the totals themselves should be equally represented.

In support of the former meaning is the fact that each element of a particular outcome can be directly related to a particular spinner (so the 1 of 1+2 can be seen as the outcome on the left-hand spinner and the 2 as connected with the right hand spinner). To omit, say, 1+2 would be to deny that the first spinner can land on 1 whilst the second lands on 2.

As with most of the gadgets, the default setting is broken; whereas there are nine possible outcomes ranging from 1+1 to 3+3, only six combinations are represented in the default workings box.

The two-dice gadget

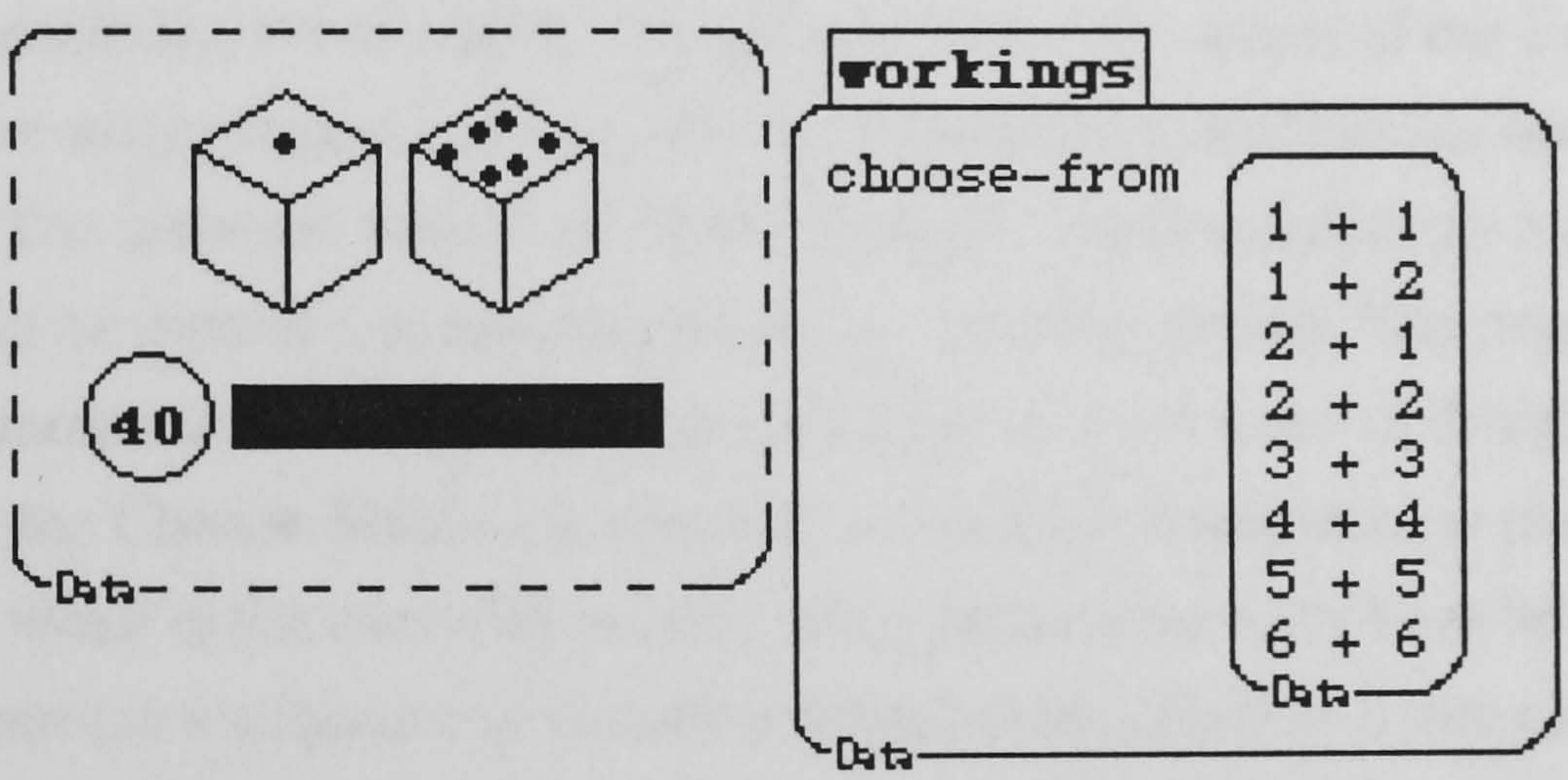

Fig. 6.5 : The workings of the two-dice gadget

The workings of the two-dice gadget are similar to the two-spinners gadget (Figure 6.5). This gadget then offers children the possibility of expressing meanings constructed from the previous gadget in a new context. The two-dice gadget is more complex in that there are 36 combinations that can be entered into the workings box in comparison to the two-spinners gadget, where one would expect only nine outcomes to be included[26].

The Frisbee gadget

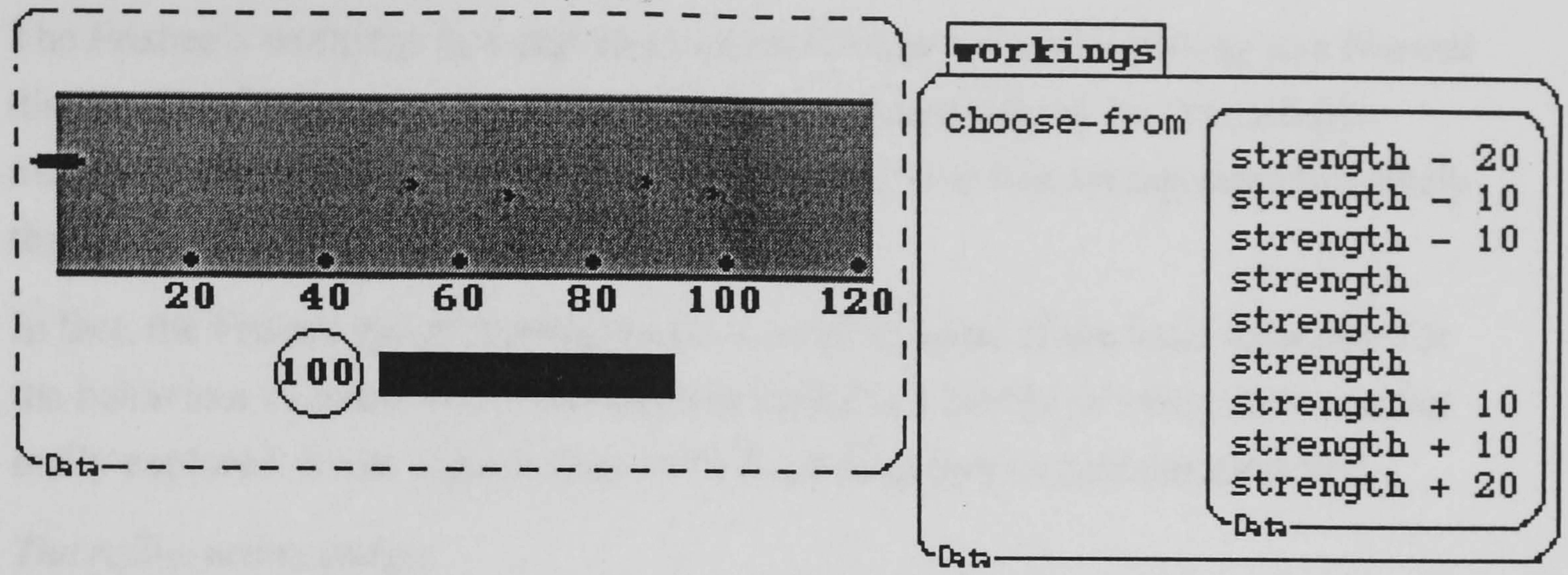

Fig. 6.6 : The workings of the Frisbee gadget

Artefacts like coins, dice and spinners are culturally well established as random devices. Potentially, we can apply a stochastic model to aspects of our lives which culturally we might regard as lying outside of the normal confines for the stochastic. The intention behind the Frisbee gadget was to incorporate a gadget which might be regarded as non-random in an everyday setting. The computer-based environment offers new opportunities here: in the context of designing gadgets for the Chance-Maker microworld, a stochastic interpretation may seem reasonable when in the everyday world such a perspective would not be used. A child may not see a sequence of distances travelled by a Frisbee in the park as random because the distance depends on many variables — the strength of the throw, the ability of the thrower, the angle of the throw, air resistance, wind forces ..... On a computer though, the programmer has to communicate somehow the throw-by-throw variability. A deterministic approach, which may seem a reasonable way to interpret Frisbee-throwing in the everyday world, is more or less impractical in virtual reality. To attempt to build into a program the many variables that could affect the performance of the Frisbee is an almost impossible task; a stochastic (modelling) approach allows many of the variables to be called *random*. On the computer therefore, formalising the apparently non-random as random can make good sense. In fact, the Chance-Maker microworld may potentially offer the child critical support in widening the range of culturally-based enterprises which are interpreted as stochastic.

The Frisbee gadget reports the distance travelled, perhaps viewed deterministically since the length of the Frisbee's journey has a clear dependence on the strength with which it is thrown. On the other hand, the child may regard such a one-to-one correspondence between strength and distance as inappropriate when a multitude of other factors help to determine the actual distance travelled.

The Frisbee's workings box expresses an approximate discrete analogy to a Normal distribution (Figure 6.6); the distance is equal in magnitude to the strength (the average distance) on most throws and distances close to this average are more likely than more extreme distances.

In fact, the Frisbee gadget proved useful to explore some of the local meanings for the behaviour of stochastic phenomena in Iteration 3 but the workings box was not really explored. Some aspects then of the Frisbee gadget remain unresearched.

The roll-a-penny gadget

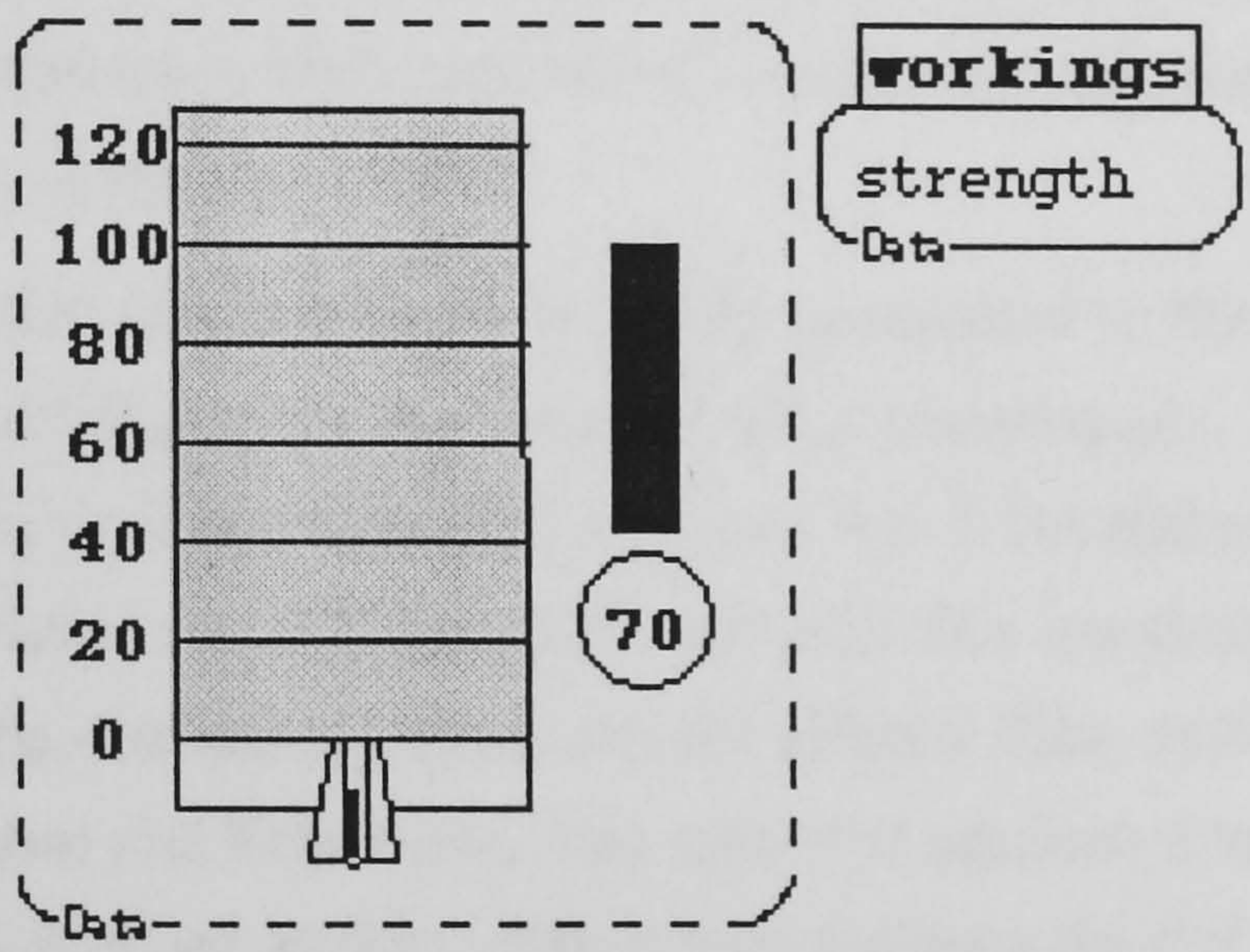

Fig. 6.7 : The workings of the roll-a-penny gadget

The Roll-A-Penny gadget has similar characteristics to the Frisbee in that it may not be seen as stochastic in nature. This gadget is designed to stand in contrast to the Frisbee gadget (and indeed to the other gadgets) by offering a non-stochastic formalism (Figure 6.7). The child who sees roll-a-penny as non-stochastic will find support in the workings box. After experience with the Frisbee gadget however, children may wish to edit the workings to make it behave more like the Frisbee.

The workings box is, for all gadgets, a non-conventional, but nevertheless formal, expression of a discrete probability distribution. Like conventional expressions for distribution, it holds explanatory power for those initiated into its meaning, but unlike conventional expressions, that meaning can be constructed through dynamic interaction with the gadget.

This section has described the nature of the new formalisms for the stochastic, in the form of workings boxes associated with a collection of gadgets. These gadgets needed to be presented in ways which would engage children. The next two sections address the issue of engagement.

6.2.2. Designing for Familiarity

A computer-based domain can seem as remote and disconnected from reality as any mathematics lecture. In a context where the child has at best a naive appreciation of the embedded mathematical ideas, a computer-based setting may appear to be completely alien. I would argue that there is a need to create intimacy, a closeness in which the structuring resources look and behave like everyday objects. A potentially alien territory is rendered more familiar by using representations drawn from the child's culture. I refer to this aspect of intimacy as *cultural familiarity.* Coins, dice, spinners, the Frisbee are not new ideas. Through cultural familiarity, the Chance-Maker microworld encourages the child to make use of intuitions constructed from experience with such objects in the everyday world applied to their doppelgängers in the microworld.

It is also important that this familiarity is deeply constituted so that the cultural artefacts behave in critical ways like their everyday counterparts. I designed the coin to have the same physical appearance of a coin of the realm and to spin much like the real thing. Similarly, the dice resembles and rolls like a real dice. The coin flips or rolls down a slope, the spinner turns and the Frisbee flies, each recognisable through its appearance and behaviour. This aspect of intimacy I term *surface familiarity.* Of course, there are huge differences between the real and the simulated worlds but, nevertheless, I claim that there is sufficient familiarity to make redundant any further explanation of what these objects are supposed to represent.

6.2.3. Designing for Purpose

One image of the design process is that of planting the new formalisms into an environment for the children to encounter. The planting process is not arbitrary; effective design involves siting the powerful idea in close proximity to the likely trajectories that trace the paths children might take in their interactions with the computer-based resources. Critical in this decision is the designer's perception of the way in which the software may be used. Thus we have to consider for what purposes the child makes use of the resources provided.

The story that is told to children in setting up their activity is therefore an important part of the design. My story was as follows. I suggested that I was in the process of programming these gadgets (in the iterative design methodology, this was indeed the case) to behave as much like the corresponding everyday object as possible. So the dice should behave like a real dice, the coin like a real coin and so on. I proposed that some gadgets were perhaps not yet working properly, and so were in need of mending. I, of course, left the children to decide what the phrases *working properly* and *mend* meant to them. In any case, it seemed to us that this activity would lead naturally to the children engaging with the workings box, offering the opportunity to witness their thinking as manifested in their on-screen actions and in their discussions.

I have presented some evidence in previous iterations that fairness was sometimes seen as an important aspect of randomness. It turns out in Chapter Seven that fairness is indeed an important and influential construct. Children typically identified a need to mend those gadgets which were not fair. This would involve building modified versions of the gadget through amendment of the workings box. The drive for fairness would often be characterised by iterative behaviour, which cycled between the observation of the effects of mending (either in terms of the appearance of the gadget or in terms of the influence on results) and further mending activity. In this sense, fairness tended to shape much of the children's activity. That mending became a legitimate and independent activity can be seen as evidence that they had taken ownership of the purpose as initially presented.

The remaining parts of this section describe the tools and controls that were made available in order to optimise the likelihood that children would construct new meanings and utilities for the stochastic behaviour of the gadgets.

6.2.4. Designing New Control Mechanisms

It was very common for children to intuit that the strength with which a gadget was activated should influence the outcome. From a mathematical point of view, strength control might be seen as a redundant control structure. Having been inculcated into mathematical discourse, the mathematician understands that the strength of the toss of a coin does not affect its outcome (except within certain limits). Such a view might question the need to include the strength control in the microworld design.

On the contrary, it was apparent to me that, if this was indeed a child's initial meaning, then they would need the opportunity to control the mechanism in exactly that way, so that they might discriminate those aspects of the gadget which the strength did control from those which it did not.

In searching for a programming solution, I drew inspiration from computer-games and the mechanism for controlling the strength of throw has already been described in Chapter Five. The principle here is that the designer must offer control of those factors which the child might see as influential, together with structures which might enable the child to discover new types of controls.

Chapter Eight will illustrate how children constructed meanings for control which involved the use of the workings box. Here we see the central idea of constructing controls. The workings box performs a dual purpose role. On the one hand it represents a new formalism of the mathematics, whilst, on the other hand, it provides a new type of control over the behaviour of the corresponding gadget. Hence, another key principle in the design of the Chance-Maker microworld is that the new controls offered are linked directly to the mathematical concepts embedded within the domain.

6.2.5. Designing for Meaning Through Use ....

These new controls enabled the use of the gadgets in ways which optimised the construction of new meanings for the stochastic. Below, I discuss the main tools which were embedded within each gadget and whose use allowed the forging of new connections between the initial meanings and the new formalisms for the stochastic. The principle of designing for meaning through use is articulated through the design ....

.... of the workings box

The workings box is designed to be changed. In its default form, it formalises how the gadget works but the central primitive, **choose-from**, is not a static expression on the screen — it can be executed, made to do something, brought to life. There are many opportunities for children to build new workings boxes by using **choose-from** in new ways. There is a real sense in which the child uses the embedded mathematics before understanding it.

It seems paradoxical that use of mathematics can precede understanding. The Chance-Maker microworld makes explicit how this conjuring trick is performed. It is unlikely that, at the outset, the workings box is seen by children as the embodiment of a piece of mathematics. If so, one might question why the microworld was being used in the first place. The workings box is seen by them more as a device for influencing the behaviour of the gadget. The workings box has utility in this context because the task is seen fundamentally as one of mending the gadgets.

As the mending process progresses, the child begins to construct new meanings for the workings box. Ultimately, for some children, it may become a mathematical structure, a formalism for probability distribution. Thus, the workings box has utility given the nature of the task from the outset, but understanding of the mathematics follows through its use in pursuing this task.

The principle here in the design of the Chance-Maker microworld is to present the new formalism as a control, which has direct relevance to and impact upon the purpose of the task.

*.... of the **repeat** primitive*

One important characteristic of a stochastic process, perhaps *the* important characteristic, lies in is its variability from trial to trial. Therefore repetition is essential. Boxer provides a number of structures for repeating an action, the simplest of which is the **repeat** primitive. In the Chance-Maker microworld, this command is placed up front, offered to the child as a helpful tool (Figure 6.8).

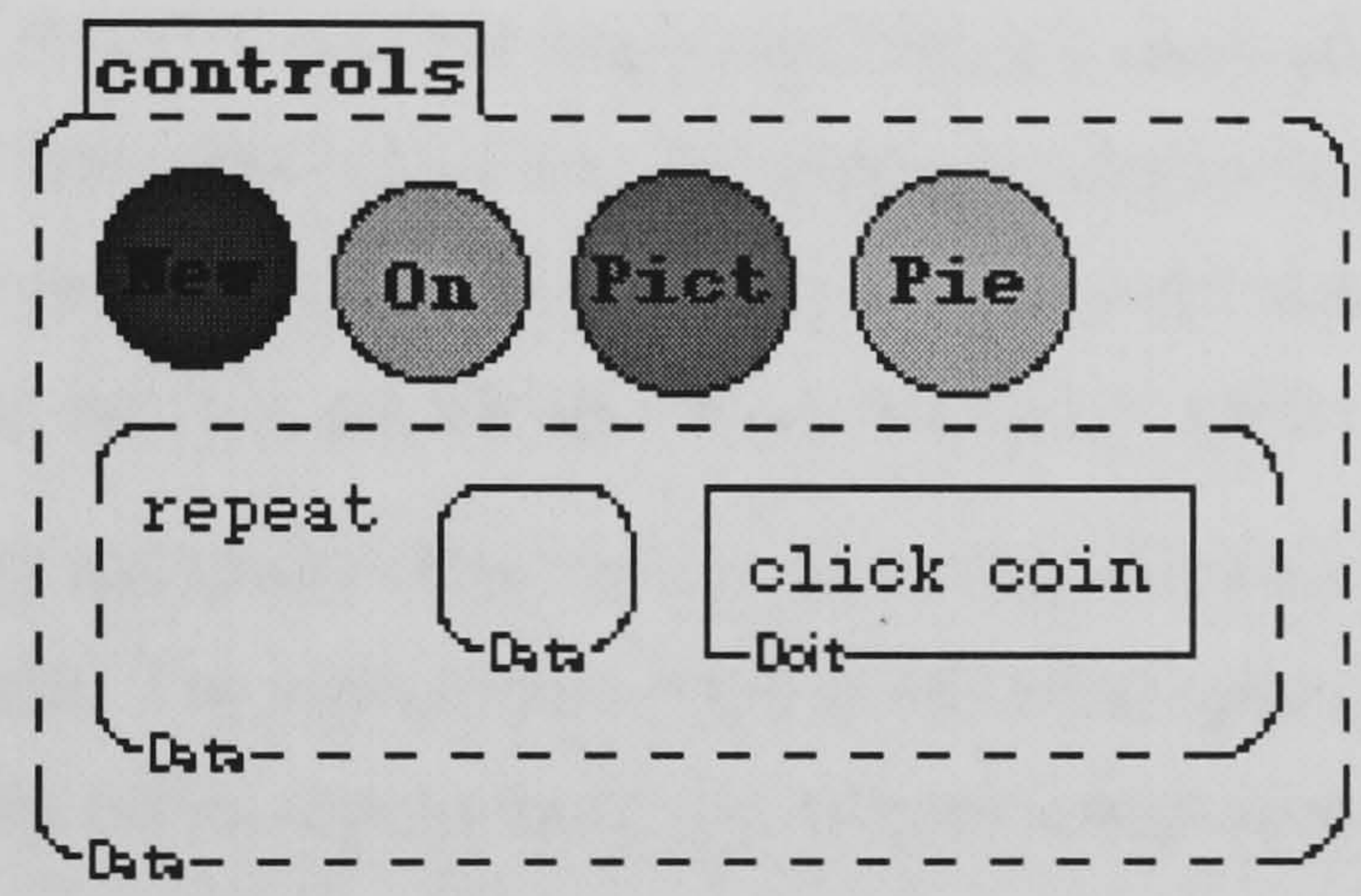

*Fig. 6.8 : The **repeat** primitive and associated buttons*

The number of repetitions is left empty; by choosing a number of trials, the child reveals something about their appreciation of the power of large numbers in stochastic processes. An essential understanding of stochastic processes is that patterns of behaviour can be observed after the aggregate of many trials even though little or no information can be deduced from small numbers of trials.

In contrast to the everyday world, the virtual reality world of the Chance-Maker microworld offers the opportunity for repetition. When large numbers of trials are used, the dynamic operation of the gadgets can become tiresome, so the graphics can be switched off (see the ON/OFF button in Figure 6.8).

Replication, the notion that an experiment can be repeated under identical conditions in order to verify a conclusion, is a special case of repetition, and is also usually impossible in everyday contexts. It was a common occurrence for children to conjecture deterministic causes of the stochastic behaviour (see Chapters Seven to Nine). One strategy for refuting such causes of the behaviour was to replicate the experiment. At top-level, a simple click on any gadget activates a single replicated trial. This *click-and-play* feature lends great affordance to replication. Replication of experiments involving repeated trials is almost as easy. The *New* button (Figure 6.8) allows the child to begin a fresh experiment in which the results box has been emptied and the number of goes set back to zero, avoiding the confounding of new and old results.

... of the results box

Randomness is likely to result in irregularity in the sequence of results. Of course, randomness could *accidentally* generate some sort of pattern, and so the child needs to be able to replicate an experiment under such circumstances simply to see if it happens again.

In other words, looking back at the history of an experiment in order to search for patterns is important, but it is equally important that any such patterns which do emerge can be tested through replication. Accidental patterns may lose their salience once they are seen to be transient. The outcome of any particular trial is reported in various ways. First let me discuss the *direct reporting of results.*

The most immediately accessible form of the reporting of results is that the gadget itself displays the result. The coin shows head or tail after spinning; the dice indicates its final value on its top surface; the spinner's arm comes to rest at its final value and so on (see the spinner in Figure 6.9). The most recent result is confirmed in a box called *result* (Figure 6.9).

When playing at top-level, the children were able to remember the most recent results and often engaged in attempts to spot patterns in short term sequences.

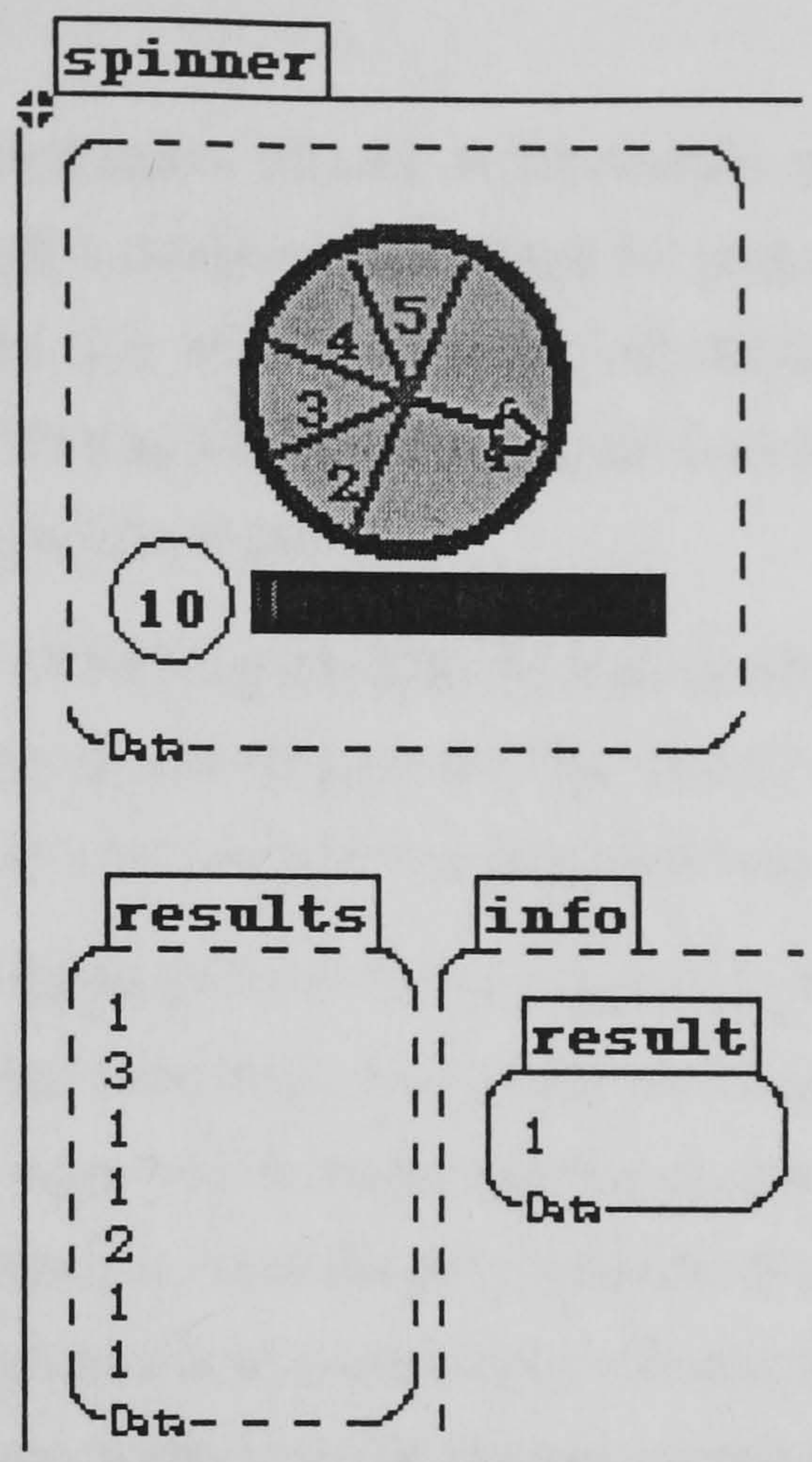

Fig. 6.9: Three ways of reporting results

The Chance-Maker microworld provides a *results* box (Figure 6.9), which simply lists all the outcomes for any particular gadget since the child last chose to empty the results box by clicking on the New button. The most sophisticated attempts to look for patterns need the list of results in order to overcome memory limitations.

The results box would enable the articulation of meanings through use. For example, children would notice what was seen as unfairness in the results (Figure 6.10). Such observations would generate new activity, involving changes to the strength or amendments to the workings box.

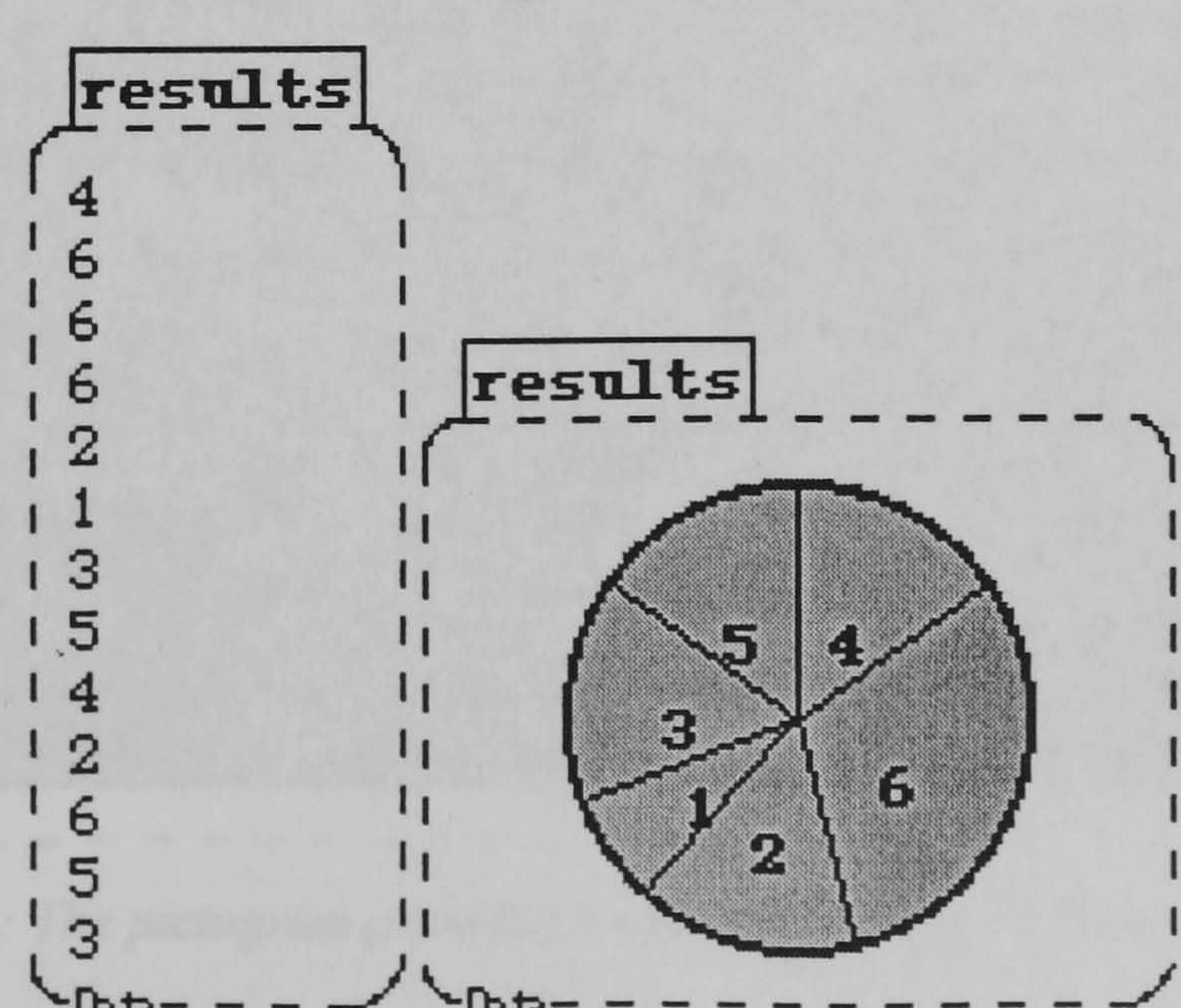

Fig. 6.10 : Unfairness can be perceived in the results box and in the pie chart

.... of the graphing tools

The results box provides a detailed history of the child's experimentation with any particular gadget. Although valuable in searching for patterns or lack of patterns, this detail is counter-productive when it comes to identifying patterns in the aggregated results. The *Pict* and *Pie* buttons (Figure 6.8) allow the child to generate pictograms and pie charts of the results.

Since this is another way of looking back at the history of the experimentation, it made sense to place the graph drawing on the flip-side of the results box. This also helped pragmatically since screen space was at a premium.

Chapters Eight and Nine illustrate how the two types of graphs played a central role in the construction of global meanings. Use of the pie chart tool enabled the construction of new meanings which connected the proportionate representation of an outcome in the workings box with the proportionate size of a sector in the pie chart. Use of the pictogram tool was particularly influential in constructing meanings for the behaviour of the totals in the two-spinners and two-dice gadget. The pictogram provided a clear image of the triangular pattern (Figure 6.11), which connects the frequency of representations of the various totals in the workings box of these gadgets to the frequency with which totals occur after a large number of trials.

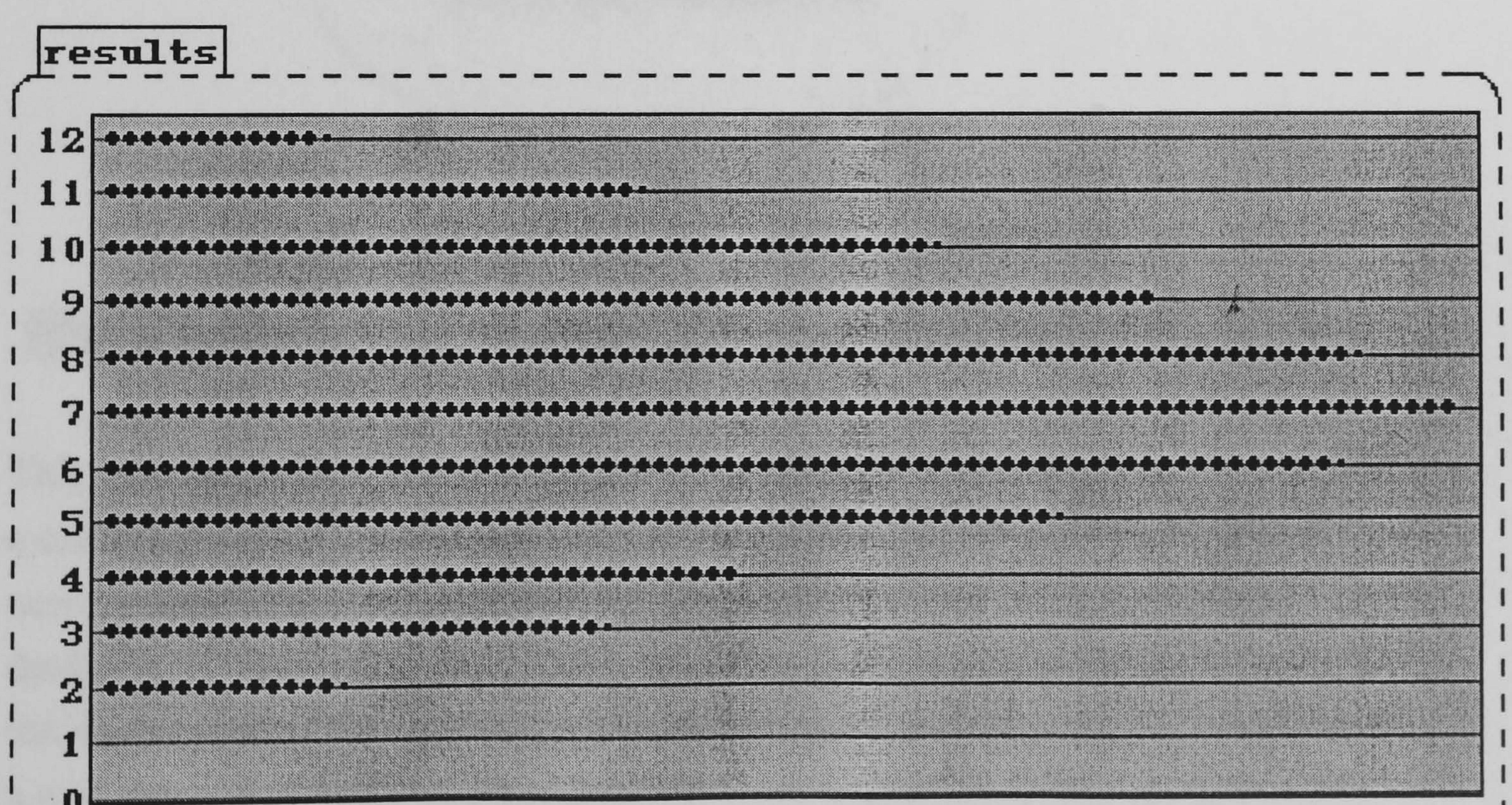

Fig. 6.11 : The pictogram provided a clear image of the triangular pattern

6.3. DISCUSSION AND SUMMARY OF CHAPTER SIX

Meanings for the Chance-Maker microworld have fallen into five categories:

(i) designing new formalisms,

(ii) designing for familiarity,

(iii) designing for purpose,

(iv) designing new control mechanisms, and

(v) designing for meaning through use.

These five meanings are closely linked. For example, the process of designing new formalisms may have begun by an epistemological analysis of the mathematical knowledge subsumed under the term, the stochastic, but it was soon shaped by pedagogic consideration of how to engage the children in using the formalisms in order to construct new meanings effectively. The relationship between these components of the design process is schematised in Figure 6.12.

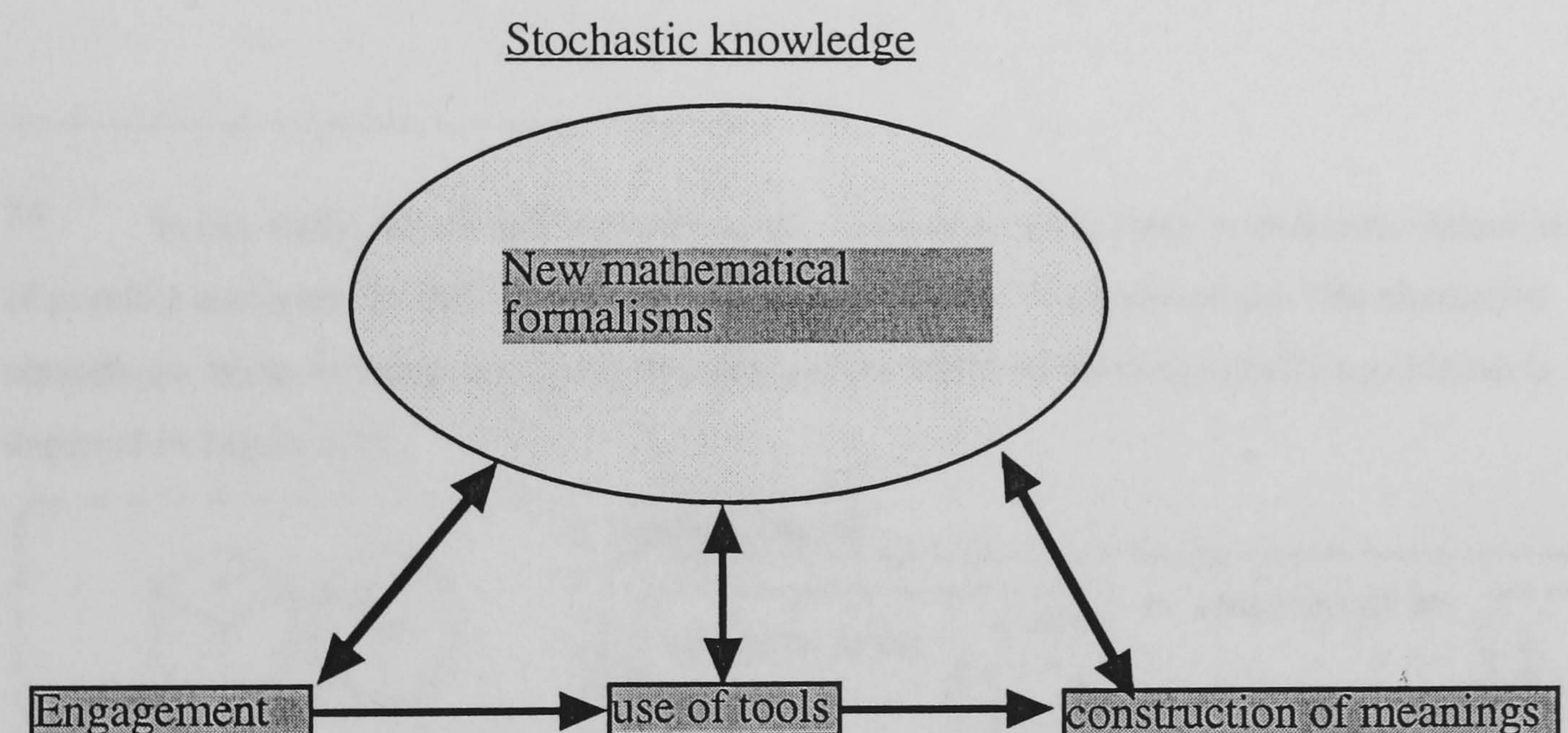

Fig. 6.12 : The design loop

This sketch illustrates how the design process is informed by consideration of how a child may pass through each of the phases in the design loop. The design will need to engage the child in activity, a stage which is supported by attention to designing for familiarity and for purpose. These pedagogic considerations re-shape the notion of what the formalisms themselves should look like, which in turn cause a reassessment of familiarity and purpose. Once engaged, a child will begin to use the tools. As with the previous stage, the tools will be informed by the nature of the mathematical formalisms but, as the children use the tools, those formalisms will themselves take on new forms in the design process. As a consequence of using the tools, the child will construct new meanings, which may not bear the intended relationship to the mathematical formalisms, in which case further re-shaping of the formalisms will again take place. The design loop recycles to its starting point since the constant renewing of the formalisms may have changed the effectiveness of the design with respect to engagement.

In the Chance-Maker microworld a key break-through was the invention of the workings box. The workings box provides the link between the formal and the informal. It acts, at one and the same time, as an encapsulation of the mathematics at the focal point of the knowledge domain in question, the stochastic, and as a control mechanism, which becomes the pivotal axis for informal exploration. It is in this sense that the Chance-Maker microworld blurs the distinction between the formal and the informal.

In the next three chapters, I will set out how these design principles were played out in practice when children came to use the Chance-Maker microworld in Iteration 3.

---

26 In this study, we see in Chapter Nine how children systematically extended the default list of possible outcomes. Indeed, several pairs managed to find all 36 combinations. One alternative amendment to the workings box of the two-dice gadget, which no children actually considered, is depicted in Figure 6.13.

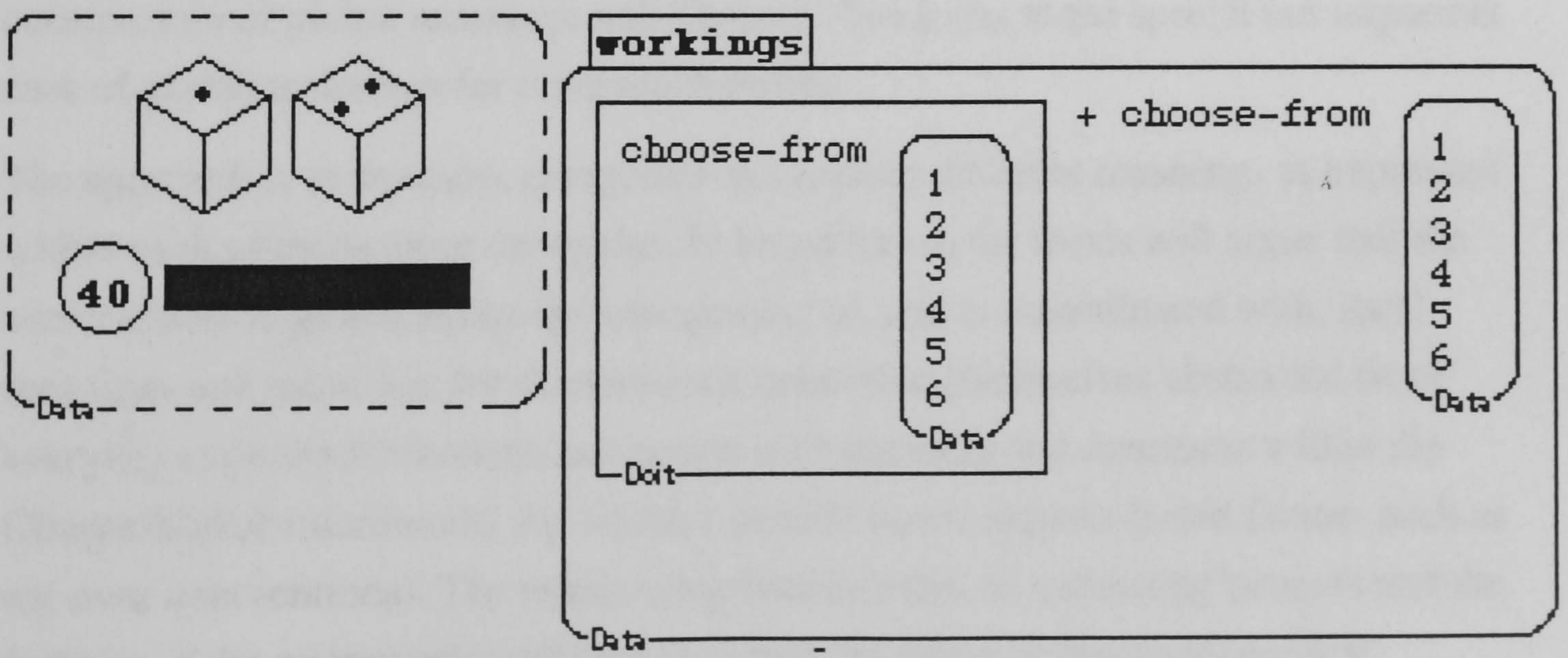

Fig. 6.13 : An alternative form for the workings of the two-dice gadget

CHAPTER SEVEN

Local Meanings *In* a Domain of Stochastic Abstraction

7.1. OVERVIEW OF CHAPTERS SEVEN TO NINE

This chapter analyses the children's tool use in Iteration 3. Whereas in Chapter Six, I placed emphasis on system design issues, meanings that I have constructed *for* the Chance-Maker microworld, I now switch the emphasis to inspect the meanings that children appear to construct when interacting *in* this environment. As I have previously intimated, these two strands are deeply connected and so, as in the Chapter Six, the children's meanings for randomness will not be discussed in isolation from aspects of the microworld design; on the contrary, throughout these chapters, I will discuss how the children's construction of meanings is shaped by the available tools and structures.

Iteration 3 drew on the insights gained from the previous iterations to analyse with increased systematicity the children's articulations of the behaviour of gadgets in the Chance-Maker microworld.

We have seen, in Chapter Five, how it is helpful to make an important distinction between local and global meanings. Chapter Seven analyses expressions of local meanings in the eight case studies of tool use. Chapter Eight focuses on construction of global meanings and Chapter Nine looks at the special but important case of global meanings for compound events.

The approach is to describe, categorise and explain different meanings as expressed within each of those three categories. In broad terms, the thesis will argue that the construction of global meanings emerges out of, and is co-ordinated with, local meanings and meanings for deterministic behaviour (themselves abstracted from everyday experience) through interaction with the tools and structures within the Chance-Maker microworld (by which I include non-computer-based factors such as my own interventions). The relationship between this co-ordinating process and the features of the microworld will also be discussed. More elaborate theoretical considerations will be postponed until the final chapter.

7.2. INTRODUCTION

The view which emerges from Chapters Five is one in which children do seem able to articulate certain types of meanings for the behaviour of stochastic phenomena. A distinction has been made between local and global meanings for the behaviour of stochastic phenomena. This chapter will look with increased systematicity at local meanings, as constructed by children during their interaction in the Chance-Maker microworld.

Before beginning that analysis, I discuss briefly in the next section the procedures adopted during the tool use phase of Iteration 3.

7.3. PROCEDURES FOR TOOL USE IN ITERATION 3

The eight pairs of children were each first introduced to the coin gadget. They were shown how to interact at top-level with the gadget by using the strength control or by clicking directly on the gadget to replicate a trial with the same strength as in the previous trial. (The detailed schedule used during the clinical interviews is given in Appendix A3.3.)

In each case, I explained that I was trying to program the gadgets to make them work properly and that their task was to tell me whether they thought that these gadgets were in fact working properly, with particular reference to whether they could predict the next outcome and whether they could control the gadget. I also explained that later I would show them some tools that they could use to mend those gadgets that appeared to be working incorrectly. They were however not yet introduced to those tools, so that initially each pair of children was required to work at top-level. My agenda here was to try to afford opportunities for observing expressions of their meanings before the computer tools began to exert significant influence.

Each pair of children was invited to play with the coin gadget and then, with no further introduction, to use the spinner and the Frisbee gadgets in that order. In each case, I asked them to try to decide whether the gadget seemed to be working properly.

After a few minutes of working with these gadgets at top level, I returned to the coin gadget, and introduced the tools hidden inside this gadget. I explained that these tools would help them to confirm or refute their ideas about whether the gadget was working properly, and, in those cases where they felt there was a problem, they could try to mend the gadget using these tools. I explained each of the tools and structures within the coin gadget, finishing with the workings box. The children then spent varying amounts of time with the coin gadget. Some saw the coin gadget as unproblematic and quickly moved onto the next gadget. Other pairs needed to spend some time, resolving issues about the coin's behaviour before being ready to try another gadget. In a similar way, different pairs spent varying amounts of time with the other gadgets, which were introduced in the order: spinner, dice, two-spinners, and two-dice. In most cases, the Frisbee and roll-a-penny gadgets were not introduced for reasons of time and complexity.

As the children worked with these gadgets, their actions on the computer and their discussions, supplemented by my probes and interventions, were recorded on videotape. The resulting case studies are accessible as plain accounts and interpreted analyses on the World Wide Web from address:
http://www.warwick.ac.uk/wie/staff/DP.htm

The next section presents, in a synthesised form, the expressions of local meanings as apparent in those case studies.

7.4. LOCAL MEANINGS

This chapter sets out a clear and unambiguous catalogue of young children's local meanings and demonstrates how the specific features in the design of the Chance-Maker microworld enabled the observation of children's meanings. The pieces of knowledge turn out to be powerful resources, which support the construction of new knowledge as will be indicated in Chapters Eight and Nine. There has been a tendency for previous writers to portray meanings for long term behaviour as misconceived (Uri Wilensky is one notable exception), emphasising the misconceptions that cause children and adults to make errors of judgement of chance, and to trivialise local meanings (perhaps because they tend to be abstracted directly from everyday experience rather than through instruction). I present these local meanings in some detail because it will become clear in later sections that they are in fact essential in the construction of new and powerful meanings.

As well as giving examples of each meaning below, I give a description of how each typically manifests itself through the features of the Chance-Maker microworld. Incidents, quoted directly from the case accounts, will be used for illustration. The reference numbers relate to the numbered paragraphs in the case accounts (see appendices at the world wide web address: http://www.warwick.ac.uk/wie/staff/DP.htm). Although these five local meanings are discussed separately, they are often closely linked. I point out such links as each is discussed.

7.4.1. Unpredictability

A common type of activity throughout the eight case studies was for children to consider whether it seemed possible to predict the next outcome of any particular gadget. Attempting to anticipate the result of the next trial frequently took place as part of a game-playing type of activity invented by the children. Gurdev and Neil, for example, engaged in predicting because they saw this as a fun activity in itself.

> Gurdev confidently predicts heads next. Neil says, "It will be tails again." It turns out to be tails. Gurdev predicts heads whereas Neil predicts tails. It is a tail again. Gurdev now predicts tails, Neil predicts heads and it is heads. I ask again if they can predict. Neil: "Yes." Gurdev: "Now I think that it's heads next, and if not tails, it's going to be heads after that, then tails."
>
> (8.2.3)

Equally often, predicting the next outcome seemed to be a natural strategy for testing whether a gadget was working properly. Generally, success in predicting was associated with non-random behaviour whereas a lack of success was seen as indicative of random behaviour. The absence of a predictive capacity in a gadget was often seen as synonymous with its property of randomness, as when Donna and Rose tried to explain the meaning of the workings box for the coin:

> Donna says, "I don't think it was saying you can just say, 'I want heads, and I want heads, and I want tails, and I want heads' .... I don't think it's going to be saying that you can choose, like the coin says, 'Right I want to be heads' or 'I want to be tails'. It doesn't really have a choice but it knows it's got to land on one of them." Rose adds, "Because you can't say, 'Oh, I want it to land on heads. it's going to come out at random."
>
> (3.7.5)

The unpredictability of a gadget seemed to be a fundamental characteristic associated with randomness. It seemed that unpredictability did not need to be explained but existed as a catch-all for those situations which could not be explained by causal factors. Unpredictable and random were interchangeable adjectives.

There is plenty of evidence in the pre-interviews and the clinical interviews to suggest that all the children in the case studies saw unpredictability as a characteristic part of the behaviour of stochastic phenomena such as coins and dice. Unpredictability is clearly a meaning that already exists within the children's internal resources *before* they use the Chance-Maker microworld. The Chance-Maker microworld though provided a window on unpredictability by affording opportunities for its articulation through three principle features: familiarity, click-and-play and replication.

(i) *Familiarity*

The process of trying to decide whether a gadget was predictable often began almost as soon as the gadgets were introduced. The gadgets appeared to lend a high degree of affordance, inviting the children to make use of them. The affordance of a gadget stemmed from its surface and cultural familiarity (see section 6.2.2). The combined affect of cultural and surface familiarities is that the child's expectations are raised as to how the gadget is expected to behave. Donna and Rose knew from everyday experience that an important feature of coins is that they you don't know what the next outcome is going to be. They did not know whether the computer's version of a coin was working properly. Therefore they were drawn into questioning the predictability of the coin gadget.

(ii) *Click-and-play*

Click-and-Play (see section 6.2.5) facilitated a games-oriented approach, which we see in the example of Neil and Gurdev's discussion of unpredictability above and which is common throughout the eight case studies. Even when the children opened up the gadgets to gain access to the tools, click-and-play was still available to encourage a return to top-level conjecturing, often associated with unpredictability, as and when the children felt it necessary. The ease with which experiments could be repeated encouraged prediction as a spontaneous playful activity.

(ii) *Replication*

It was often the case that causal factors were conjectured as the reason for unpredictability. There was then a need to standardise that alleged causal factor to find whether the gadget was still unpredictable.

In the following incident, Ray and Luke have just begun to use the coin. They conjectured that the strength controlled the outcome. They were able to test this conjecture by replicating the experiment with the same strength,

> They begin to use the coin. Almost immediately Ray observes that he thought the outcome was going to be the same. Ray explains that this was because he flipped the coin with 100% both times. He tries with 100% again. He gets head again and says, "It does come out the same." Luke tries 100% and also gets head and agrees with Ray that 100% always gives a head. Then Luke tries 50% and gets a tail, and says, "If we do it 50%, it will .... (waits for coin to finish and it lands on a tail) ...land on a tail." He tries to do it on 50% again, but gets a head. Ray says, "50% is always different." Luke agrees. In fact, the next 50% does come out as a tail, confirming their thinking.
>
> (2.2.1)

At this point, the vicissitudes of randomness had led Ray and Luke to continue believing that the strength was controlling the result. The example does give a clear image of how the ability to replicate in the Chance-Maker microworld, by simply clicking directly on the gadget (see Figure 7.2), offers opportunities for conjecturing predictability and testing that conjecture.

7.4.2. Unsteerability

Two types of control were articulated at various times by the children in the eight case studies. Later I consider the case where control was regarded as lying within the computer. Here though, I focus on the exertion of control through the degree of strength with which the gadget was activated. I term this type of control, *steerability*.

Unsteerability was frequently articulated as a local meaning for randomness. For example, when Steve and Richard were asked whether they could predict the spinner's next result, Richard replied by referring to the strength control:

> Richard continues, "If you change strengths, I think you can, because if it starts from there, if you just do a few numbers, like." Richard tries strength 10 and got a 1. Then the same strength gave 4. Steve says, "Definitely random." Richard says, "Yes."
>
> (4.3.2)

Richard believed that it was possible to predict the spinner's outcome, as he saw the result as a function of the strength. A simple test involved replicating the experiment with the same strength. When a different result was achieved, Richard agreed with Steve that the spinner was random. Apparent steerability had led him to believe that the spinner was not random but then the lack of such control convinced him that the spinner was in fact random.

Lee also saw the spinner as not random because of steerability:

> "Basically you can control the strength .... so I don't think it's random."
>
> (5.3.2)

In both these incidents, and many more across the eight case studies, unsteerability was used as a description of a gadget's behaviour. If the gadget could not be controlled then it was random as Lee so neatly articulated.

The fundamentality of unsteerability is also seen in the way it is often used to 'explain' other meanings for randomness. Very frequently, children associated unpredictability with unsteerability. On such occasions, unpredictability was usually seen as the outcome of uncontrolled input. The following incident is typical of how predictability and steerability were linked. On this occasion, Terry and Joe drew the conclusion that the roll-a-penny gadget was not random:

> They play with roll-a-penny. Terry observes: "It's landing on the same number as the strength." Terry says you can control it and it is predictable. Joe agrees. They say that this gadget is not random.
>
> (1.5.1)

The pre-interviews and the clinical interviews provided compelling evidence that unsteerability was a powerful discriminator of stochastic behaviour. The meaning of unsteerability for the behaviour of stochastic phenomena appeared to have been abstracted from prior everyday experiences. The Chance-Maker microworld afforded opportunities for the articulation of unsteerability through two principle features: the strength control (see section 6.2.4) and cultural familiarity (section 6.2.2).

(i) *The strength control*

From the outset, the children engaged with the gadgets through direct manipulation of the strength control. There is much evidence in the case studies of children coming to believe in the randomness of the gadgets by observing that they were unable to control short term behaviour, and that they needed to separate control of strength from control of distribution.

(ii) *Cultural familiarity*

The notion of cultural familiarity has been discussed in its links with the unpredictability meaning. It reappears here because the explanation of unpredictability was sometimes connected with everyday experiences of causal factors. Luke and Steve in the following incident connect the partial control of the Frisbee gadget with the partial control of a real Frisbee.

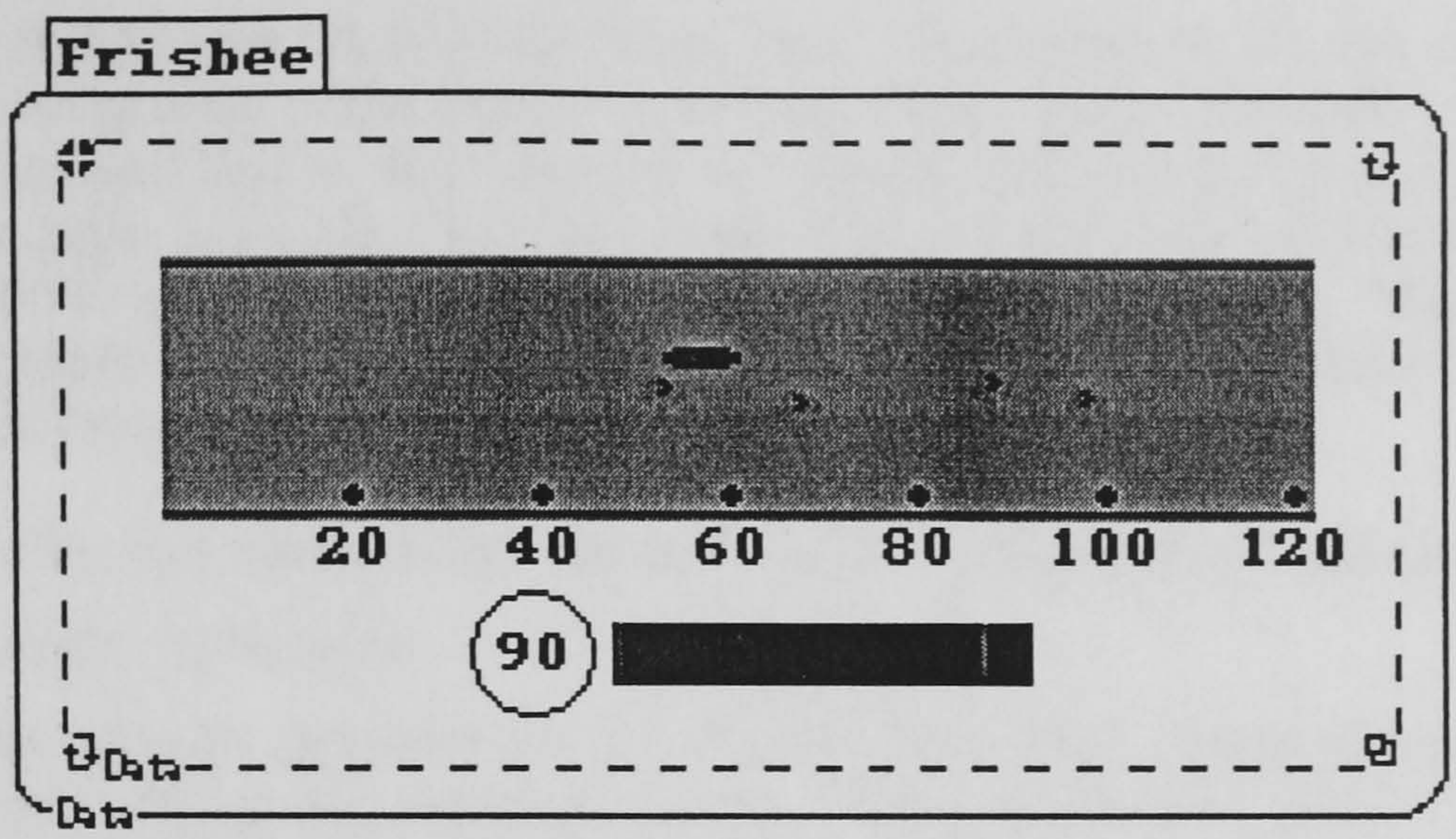

Fig. 7.1 : The Frisbee gadget in flight

The unsteerable aspect of the computer's gadget is described in terms of real factors like the wind.

> Luke and Ray say you can not control the Frisbee and that it isn't predictable. Luke says, "You can't really really predict because of the wind, same thing. And on the computer, if we done it at 60 and it either went to 80 or 40 or around 60." Ray says, "It always goes around. Like if we done 100, it would be either 80 or between 100 and 120, between 80 and 100." Ray confirms this by clicking on the Frisbee and predicting the range that it will go. Luke agrees that what Ray said is true. They both think it is random. When I ask why, Luke says, "Because of the wind again, it could blow it anywhere." Ray says, "Might just throw it at a tilt, so it won't go as fast or you could do it wrong so it would go in a circle."
>
> (2.4.2)

The wind can not be controlled by the Frisbee thrower and so the outcome is not (entirely) predictable; the Frisbee gadget is not controlled (entirely) by the strength control, and so the result is also not precisely predictable. The cultural familiarity of throwing Frisbees made the behaviour of the Frisbee gadget more intuitively acceptable.

7.4.3. Irregularity of Results

A fairly common activity in the eight case studies was for the children to try to identify regular patterns in sequences of results. This behaviour was often linked closely to predicting so that patterns were conjectured on the basis of past results and then used to make predictions, which were tested by further trials.

For example, Neil and Gurdev, who seemed especially interested in pattern-spotting activity, were playing with the coin at top-level when this typical incident occurred:

> I ask if they can control the coin. Neil: "Sometimes, it's the same. It's a sequence. sometimes." Gurdev: "Just say it was 60%, well, I've noticed that it was like just say heads first, then it goes heads again, tails, tails, heads, heads, tails, tails." I ask if this is just when it is 60% to which Gurdev says, "Generally." I clarify, "So you think there is a pattern to what it does?" Gurdev: "Yes." Neil: "Yes, I think there is as well."

Later, when they had been shown the tools inside the coin, Neil and Gurdev engaged in similar behaviour.

> As the results develop in the results box, Neil reads them out, clearly looking for a pattern. Neil: "Tail, head, tail." Neil: "Tail, head, tail, tail." Gurdev joins in this activity by predicting. Neil: "Tail, head, tail, tail, tail, head." Neil explains, "What I am doing is just putting it at different percentages and seeing what the results are going to be. To see if they go in a pattern or not." Gurdev suggests, "If you try it the same, there might be a pattern." Neil continues his pattern searching, "So far it's gone tail, head, tail, tail, tail, head, tail." He predicts a head but it is a tail. Neil: "So that's tail, head, tail, tail, tail, head, tail, tail. It could go tail, tail, head, tail, tail, like that." After 12 throws, Neil reads out again, "So that's tail, head, tail, tail, tail, head, tail, tail, head, tail, tail, tail. If it's a head now then it will be a miracle." It is a head. Neil says, "It's going in a pattern .... It's going tail, head, tail, tail, tail, head, tail, tail, head, tail, tail, tail, head." Gurdev says, "It isn't much of a pattern."
>
> (8.7.1)

Irregularity seemed to be an important meaning for randomness. Results from activity in the Chance-Maker microworld are stored in a results box. This facility was often used by the children in the eight case studies who would look back at these results in attempts to identify patterns.

The pre-interviews and the clinical interviews provided strong evidence that irregularity was frequently used as a discriminator of stochastic behaviour. The irregularity meaning for the behaviour of stochastic phenomena was already abstracted from everyday experiences. The Chance-Maker microworld afforded opportunities for the articulation of this meaning through webbing with two main features: direct reporting of results and the results box (see section 6.2.5). The most recent result was always available to the children as part of the appearance of the gadget itself as well as in a box which contained that result. When memories of recent results began to fail as the list became longer, the children could refer to the results box to search for sequences.

7.4.4. Fairness

The pre-interviews showed that fairness is, for many of the children in the eight case studies, a defining characteristic of randomness. This meaning of fairness was also articulated in the Chance-Maker microworld. For example, Cathy and Lynn decided that the spinner was not random because of a lack of fairness in its physical appearance.

> I ask if it is random. They both say, "No." Lynn explains this by saying, "It's not even. The 1's the same size as the 2, 3, 4, and 5 put together." Cathy: "Basically the same because like it's unfair because the 1 is bigger than the 5 and like the 3 and 4."
>
> (7.3.2)

The appearance of the gadget was not obvious in the case of the dice. In the following incident though, Ray believed that fairness in the workings box was a requisite property of a random dice:

> Ray says, "I think every dice is random, apart from if you have two 1's." Luke says, "Yes, apart from if you have two of something....along there (looking at the workings box) it has got three 6's. It's got 1, 2, 3, 4, 5 then it's got three 6's." I ask, "What do you think that's going to do?" Luke: "It may make it easier to get 6's."
>
> (2.8.1)

A third way in which fairness was articulated was through the results themselves. Anne was concerned at a lack of fairness in the first few results of the coin gadget (before she had access to the workings box).

> In fact it is a head. Then 100% strength gives a head, and then head again, and then head again. Anne says, "It's a bit unfair because it keeps going to the queen *(i.e. heads)*." They try 100% again and it gives a tail, and then head. Anne says, "Most of the time it goes to a queen, er heads."
>
> (6.2.1)

Fairness often became a sub-goal for activity which shaped the interactions with the Chance-Maker microworld structures and tools. A typical incident arose out of my intervention, which asked Donna and Rose whether the spinner was fair. They amended the workings box which led to them making connections between the fairness of the workings box and the fairness of the spinner's appearance. Later the activity moved on to consider the fairness of the results.

> Meanwhile Donna has edited the workings box to read **choose-from [1 2 3 4 5]**. She explains to Rose, "It had four 1's, and I took off three of them, so then it made it split up equally, and we are going to see what the chart is going to look like." I ask them what they think it will look like. Donna: "I think it's going to divided equally." The spinner changes to be equally divided. Donna says, "That's basically what I think it's going to look like." I clarify, "Right, so you think the pie chart is going to look like the spinner." Rose says, "Maybe." Donna says, "I think it will look roughly like that."
>
> (3.8.6)

It is clear from both the pre-interviews and the clinical interviews that many of these children saw fairness as a defining characteristic of randomness. These meanings existed before their interactions with the Chance-Maker microworld, presumably abstracted from their games and role-playing everyday activity. The Chance-Maker microworld acted as a window on these meanings by providing webbing opportunities with four features: unfairness in the gadget's appearance, unfairness in the workings box, unfairness in the results and in the pictogram or pie chart (see section 6.2.5). Each of these features provoked extended engagement with the gadgets, usually with associated attempts at mending.

7.4.5. Computer-in-Control.

The computer-in-control local meaning was more specific to the Chance -Maker microworld, not drawn directly from everyday experience. Some children did not, at least initially, regard the gadget as random because it had been programmed. Rebecca, for example, rejected the coin's randomness because there must be some sort of pattern in the way that the computer was programmed, even though she had not found such a pattern.

> I ask if the coin is random. Rebecca: "Not really ... it's probably been programmed to do it, in a loop." Anne says she doesn't know. I ask Rebecca what she meant by 'in a loop'. Rebecca: "Well, it's programmed to do heads, then maybe heads again and then tails." I clarify, "In some sort of pattern?" Rebecca: "Yes."
>
> (6.2.2)

Similarly, Lee rejected the notion that the coin was random because it had been programmed.

> I ask them if the coin is random. Lee says, "I don't think so because someone has programmed it to do that .... they've put in, if you do strength 40, first time it will do heads, second time it will do heads, but, if you do strength 40, on the third time it will be tails."
>
> (5.2.1)

In the above incidents, no pattern had been located but the fact that such a pattern must exist in a programmed environment was sufficient to reject randomness. Lee used the same sort of argument to identify that the roll-a-penny gadget was not random but this time a pattern was obvious.

> I ask if the distance that the coin is rolling is random. Lee: "No, because someone has programmed it to say, if it is strength 100, it goes on 100, so it isn't random, no."
>
> (5.5.1)

Later Lee, encouraged by his partner's observations, began to connect the computer control with factors in the behaviour of everyday phenomena. He began to regard the programming of variability as akin to factors like the wind, which influence the flight of a Frisbee but are outside of personal control.

> "Because the person programmed it, they go at different types of distances, it's like the wind that's changing." Lenny thinks it is in some sense behaving like a real Frisbee. Lee: "It's like the wind is blowing it at different speeds."
>
> (5.4.1)

This linking with everyday phenomena sometimes helped children across what might otherwise have been a barrier hindering further effective use of the microworld. Another route across this barrier was to recognise that the task was to simulate the behaviour of everyday phenomena. In this view, it did not matter that everyday randomness and computer randomness were different ontologically as long as the effects on the computer matched those in everyday. Anne and Rebecca began to view the computer's randomness in this way in the following incident.

> I ask if they think this coin is working properly. They think not. Anne: "On the computer, it can choose which one it wants. You can't really choose." Rebecca: "You can't decide yourself whether it's going to be heads or tails." I explain again that I am trying to programme the computer so that it behaves as much as possible like a real coin, so that we can't distinguish the results from a real coin. Rebecca: "I suppose if you used the **choose-from**, you could probably choose it by clicking on that and telling it to throw, it would probably throw the one that you chose."
>
> (6.7.7)

By making connections with everyday phenomena or simulations of randomness, the children were able to adopt a modelling perspective for randomness which in turn would open up possibilities for the construction of new global meanings, related to the long term behaviour of stochastic phenomena.

I frequently asked the children what they understood by the workings box. Such probes prompted some fascinating descriptions of how the computer exerted its control. These images of the mechanism by which the workings box exerted its control can be categorised in two ways, each associated with one of the above two incidents. On the one hand, children attempted to explain the workings box using metaphor, drawing on familiar everyday phenomena, whereas, on the other hand, they appealed directly to randomness through reference to local meanings for various attributes of short term behaviour.

Below I give some typical incidents in which children made use of metaphor to describe how the workings box operated.

> Rebecca: "I'm not too sure, it could just sort of send like a cursor going bleeping from one to the other, and tell it to stop at a random time." I ask, "So you think it is choosing randomly?" Rebecca: "Sort of. Probably."
>
> (6.7.5)
>
> Cathy: "Behind like that could be highlighted (pointing to the top head in the workings box) and it go tuh, tuh, tuh, tuh, tuh, tuh *(oscillating the mouse up and down between the head and the tail in the workings box as she makes these sounds)* and stops."
>
> (7.2.2)
>
> I ask what the workings are telling the computer to do. Donna: "It can't choose itself numbers 12 or 8 or something. It can only be between 1 and 6 .... it would just topple over and know it could only land on one of those .... randomly."
>
> (3.9.11)
>
> "What do you think the workings are telling the computer to do?" Neil: "Well, really, it just makes it look like a flick. But it's just choosing a head or a tail." I ask, "How do you think it is choosing whether it is a head or a tail?" Neil: "I don't know. Probably, it just, I don't know, flicks it."
>
> (8.7.4)
>
> Luke: "Choose from heads or tails. Just when it flicks it, choose one of them for it to land on." I persist, "And how do you think it chooses?" Luke: "Well, it uses a coin *(laughing)*." Ray: "Just by random." I say, "As if it were using a coin?" Luke says: "As if it were using a coin, yes."
>
> (2.6.6)
>
> I ask what they think the workings are telling the computer to do. Richard says, "Choose from both." Steve agrees. I ask how it chooses. Richard says, "Just tosses it and catches it." Steve agrees, "Yes, it just tosses it and catches it and whatever it lands on."
>
> (4.7.4)

On other occasions, some children (sometimes the same children) simply appealed to other local meanings. For example, Rebecca expected, at least initially, that the workings would result in a pattern.

> I first ask about the workings. Rebecca: "Put it on this number this time and maybe that number the next time." I say, "So you think there is a pattern to the results?" Anne: "Yes." Rebecca: "Maybe."
>
> (6.8.8)

Similarly, Terry refers to unpredictability or irregularity.

> I ask them what the workings box tells the computer to do. Terry says that the workings box is to make it so it is random. "So that it doesn't choose heads all the time or tails all the time."
>
> (1.7.4)

The Chance-Maker microworld provided webbing opportunities for the children's articulation of the computer-in-control meaning through two features: cultural familiarity (see section 6.2.2) and the **choose-from** primitive (see section 6.2.5).

(i) *Cultural familiarity*

There were several examples of children explaining the variable behaviour of gadgets such as the Frisbee in terms of imagined factors like the wind which blew the gadget away from its otherwise pre-determined course. Factors like the wind were envisaged as being programmed into the computer. Since the detailed mechanism by which this happened was unknown to the children, or randomness itself could not be further broken down into constituent explanatory components, they referred to the mechanism as if it *were* the everyday phenomena. The fact that the children were able to make these close associations between a gadget on the computer and an everyday phenomenon is, in my view, testament to the familiarity of those computer gadgets.

(ii) *The **choose-from** primitive*

Another strategy for dealing with the variation in gadgets' behaviour was to attribute that behaviour to primitives, particularly the **choose-from** primitive. The images of how the computer exerted this control through the **choose-from** primitives were idiosyncratic. The children had no language with which to express this process so they either used metaphoric comparisons with everyday devices as above, or they simply appealed to meanings for randomness like irregularity or unsteerability.

7.4.6. Summary of Incidents Involving the Five Local Meanings

The examples given above are representative of many such incidents drawn from across the eight case-studies. The matrix in Table 7.1 provides a cross-reference to these and other incidents involving these five local meanings (the references in the final column relate to the numbered paragraphs in the case accounts – see appendices at the world wide web address: http://www.warwick.ac.uk/wie/staff/DP.htm).

	Unpredictability	Unsteer-ability	Irregularity	Fairness	Computer-in-control
Terry & Joe	1.2.2; 1.5.1; 1.3.1-2	1.4.2-3; 1.5.1			1.7.4
Ray & Luke	2.2.1; 2.3.2; 2.4.2; 2.8.1	2.3.2; 2.4.2	2.6.1; 2.6.5	2.3.2; 2.8.1	2.6.6
Donna & Rose	3.2.1; 3.3.2; 3.4.2; 3.7.5	3.3.1; 3.4.2; 3.7.5	3.2.2; 3.7.2; 3.9.7	3.3.2; 3.8.6	3.3.1; 3.9.11
Steve & Richard		4.3.2	4.2.3; 4.8.3; 4.8.4		4.7.4
Lee & Lenny		5.3.2; 5.4.1	5.2.1; 5.3.2; 5.6.2; 5.7.3	5.3.1	5.2.1; 5.4.1; 5.5.1; 5.8.20
Anne & Rebecca	6.3.2; 6.7.2	6.3.2; 6.7.2	6.2.2; 6.7.1; 6.7.5	6.2.1	6.2.2; 6.4.3; 6.7.5; 6.7.7; 6.8.8
Cathy & Lynn	7.2.1-2; 7.4.2	7.2.1; 7.3.1; 7.4.2	7.7.1; 7.7.2	7.3.2	7.2.2
Neil & Gurdev	8.2.6; 8.3.4; 8.7.5	8.2.5; 8.7.5; 8.8.14	8.2.2; 8.2.3-5; 8.7.1;	8.3.1	8.7.4

Table 7.1 : Occurrence of local meanings for the behaviour of stochastic phenomena

Table 7.1 gives an impression that the five meanings were widely used by all eight pairs of children. It can be seen from the table that there was considerable variation in which meaning was used by the same child, an issue which is the focus of my attention in the next section.

7.4.7. Variations in Individual's Use of Meanings

There was considerable variation in which particular local meaning for random behaviour was cued even for the same pair of children. Consider the case of Donna and Rose, which will serve to illustrate this variation, which was observed in all of the children in the eight case studies.

Donna and Rose had both used unpredictability as a key discriminator of randomness in their pre-interviews. They had also referred frequently to fairness. Rose perhaps was more inclined to view fairness as an attribute of how the device was made, referring explicitly to the sides and surface of the dice itself, whereas Donna paid more attention to the outcomes of a few throws.

Almost immediately when the two girls begin to use the first gadget, the coin, Rose commented:

> Well, it's not going to come out as you think it is. It's going to come out any old how.
>
> (3.2.1)

Donna agreed with this view which articulated unpredictability as a meaning for random behaviour, as observed in their pre-interviews.

In her pre-interview, Donna had been confident that the uniform spinner had been random and that all the numbers were equally easy to obtain but she had been less sure about the non-uniform spinner. After some consideration, she had decided that the non-uniform spinner was not random because it was not fair, the sectors being unequal in size.

In the work with the computer's spinner, Donna began by conjecturing that the strength controlled the outcome but, after seeing some variation in the results, she changed her mind, presumably noticing the unpredictability (3.3.1). Only a few moments later, Donna accepted that the spinner was random.

> Donna says that it is random because you don't know what it is going to come out as. I ask, "Do you think it is fair?" Donna says. "Yes." but Rose interjects, "No, not all of it's fair, because you've got the 1 — it has half of the board ...." Donna says she was thinking about the spinner rather than the numbers and that she does not think it is fair as far as the numbers are concerned.
>
> (3.3.2)

When she said that the spinner was random, she was also thinking of the spinner as fair. Rose's interjection caused her to reflect that the spinner was not in fact fair as she then attended to the numbers on the dice. It is unclear what Donna had been thinking about in her first response but one interpretation is that her focus on unpredictability had led her to think of fairness in terms of not being able to say in advance what the outcome would be (thus a player would have no advantage over another player in a game based on predicting the outcome). Whether this interpretation is correct or not is less important than the message that Donna's focus on unpredictability allowed her to see the spinner as random and to ignore issues of fairness, which might in fact have been quite transparent, and certainly were to Rose.

At the same time, Rose's own meaning-making is of interest. She understood in the pre-interview that some numbers on the non-uniform spinner would appear more frequently, but she nevertheless saw the spinner as random as "it might come out with like, any number, however hard you flick it." The lack of fairness was not an issue for her on this occasion.

> I ask them whether they can control the spinner. Donna says, "No, I don't think so." Rose says, "I don't think you can control it because you aren't actually flicking it yourself. The computer's actually doing it." I ask whether they think it is random. Rose says she is not sure.
>
> (3.3.2)

Whereas unpredictability had been the key discriminator of randomness for Rose in her pre-interview, we see here that a computer-in-control meaning of the spinner's behaviour was being cued by the computer setting.

On many occasions, different meanings of randomness were cued without presenting anything problematical. Unsteerability was articulated alongside unpredictability; the fact that a situation which is not controlled is often not predictable allowed these meanings to be used interchangeably.

> I ask them if it they can control the Frisbee. Donna: "Well, you can control it within say about 40." Rose: "I don't think you can control it because the wind it comes all different directions and the wind might take it another way and it might bring it back so it might not go as far as you think it's going to go." I ask them if they can predict how far it's going to go. Donna: "You can say within a range of 20 or 40, because so far what we've had on the computer is that it you do 40 it lands between 20 and 60." Rose: "Well you might be able to predict but I don't think you can really because it might not always land where you think it's going to land."
>
> (3.4.3)

Similarly, irregularity in historical results was seen as being almost isomorphic to unpredictability. Thus later when using the coin again:

> They look at the results box and ask them what information they get by looking at them. Rose says, "They don't go like head, tail, head, tail. They come out random." I say, "So, there's no pattern to them." Donna agrees.
>
> (3.7.2)

The interchangeability of these meanings suggests that for Donna and Rose, and in fact for the other children in the study, these meanings have been clustered through experience into a structure, in which there are connections between the various meanings.

This flexibility of response can be seen as rather powerful since, if one meaning is not appropriate, then another can be brought in as a sense-making tool. This sort of flexibility was apparent in all the children's work with the Chance-Maker microworld, though I have chosen to restrict the examples to Donna and Rose merely for illustrative purposes. Sometimes however, the different meanings would appear to us to be in contradiction. The next section discusses how these contradictory situations were not always seen as problematical by the children and, when they were, how they tried to co-ordinate them.

7.4.8. Dealing with Contradictory Local Meanings

There are many occasions when local meanings appear, at least from our point of view, to be contradictory to each other. How does a child, for example, construct meaning for the behaviour of a spinner which is regarded as neither controllable nor predictable and yet is, by default, set so that the sector representing 1 is greater than the other sectors, and so might be seen as unfair. One might expect a child who holds local meanings of unpredictability and fairness to encounter some difficulty in explaining or describing the spinner. Or how does a child regard the Frisbee which may be seen as steerable (broadly) but not entirely predictable (at a precise level)?

Such potential contradictions were dealt with through various strategies, labelled types A to E.

There is no contradiction(Type A)

The contradiction was often dealt with by simply adopting one side or the other without apparent concern. For example, Lynn believed that she was not able to control the outcome of the spinner but she also knew that it was unfair.

> Lynn: "No, I don't think you can control it .... because you can't kind of really control it with the computer unless you, I don't know...." Cathy interjects, "Cheat." Lynn continues, "You can't cheat with the computer. No, I don't think you can because, even though you've more chance of getting a 1, you could still get a 2, 3, 4 or 5. You don't really know." I ask why you have more chance of getting a 1. Lynn: "Because it's bigger. It's the same size as the 2, 3, 4 and 5 *(put together)*." Cathy agrees with this, and explains, "Well, like I said that it's like because the 1's bigger, you're more likely to get a 1 but you could get the others."
>
> I ask if they can predict what is going to come next. They both say, "No." I ask if it is random. They both say, "No." Lynn explains this by saying, "It's not even. The 1's the same size as the 2, 3, 4, and 5 put together."
>
> (7.3.1-2)

Lynn simply decided that fairness is what matters and paid no attention to the issue of control, even though on other occasions, control was foremost in her thinking. Here we see a clear example of how the Chance-Maker microworld acted as a window on Lynn's local meanings. First, I was able to observe an unsteerability meaning through her actions with the strength control. Then, I could observe Lynn's fairness meaning through discussions prompted by the appearance of the spinner. The switch in her attention from unsteerability to fairness was made evident by the careful design of tools which allowed Lynn to articulate her various local meanings, and so make transparent the surprising lack of contradiction between these meanings in Lynn's sense-making activity.

The fact that so often children were able to deal with such situations as if they were not problematical suggests that the contradiction was not apparent or at least not felt to be problematical. In fact, the contradiction is often more our construct than that of children; for them, the contradiction did not necessarily exist. Randomness can be articulated in different ways and it was not usually necessary for it to be articulated in more than one way simultaneously. Different meanings were activated or cued by different circumstances without there being any apparent contradiction. In the following incident, Steve and Richard were using the spinner and were finding that they could not control the outcome.

> I clarify, "You can't predict. You don't know what it's going to be?" Steve: "No." Richard says, "If it starts from the top every time, you can just use different strengths ..." Steve interjects, "No, actually I don't think you can predict it, because it's got sort of like a diagonal line *(referring to the line which separates half the spinner marked 1 from the other half of the spinner)*." Richard continues, "If you change strengths, I think you can, because if it starts from there, if you just do a few numbers, like." Richard tries strength 10 and gets a 1. Then the same strength gives 4. Steve says, "Definitely random." Richard says, "Yes."
>
> (4.3.2)

Half of the spinner was allocated to the outcome, 1, and the rest was shared between 2, 3, 4 and 5. The lack of fairness in the spinner was not cued even though the children made explicit mention of the uneven sharing between the different outcomes. In stark contrast, in his pre-interview, Richard had been convinced that the non-uniform spinner was not random because of its lack of fairness. We might suppose that the difference between the experimental situation and the interview for Richard was that, on the computer, control issues were much more immediate. Richard was able to vary the strength in carefully organised ways. Such fineness of control was impossible with the real spinner in the pre-interview. We can therefore account for Richard's different responses in terms of the specificities of the situation, which cued distinct meanings.

Below I continue this story to illustrate how activity within the Chance-Maker microworld can support the co-ordination of different meanings.

The domain of randomness is modified (Type B)

Steve and Richard had then been unaware of any contradiction, though later, when they again used the spinner gadget, Steve adopted a new way of articulating the behaviour of the spinner. He separated out 1 as a special case,

> Steve adds, "I think it probably is, but you've got a little but more chance of... it's random out of 2, 3, 4 and 5, but with 1 you are probably going to get that more. Well, it depends on how hard you swing it, spin it."
>
> (4.8.5)

The strategy, labelled Bi in Table 7.2, involves limiting the domain over which the randomness applies. Terry and Joe dealt with the spinner in exactly the same way as Steve and Richard by referring to it as being "random apart from 1" (1.8.3).

Contradictory behaviour also arose when children found that they could identify patterns in results but that these patterns then disappeared. Often this was simply regarded as part of the nature of randomness and so was not problematical. Occasionally, some children were puzzled by this phenomenon, usually because they paid more attention to the actual results (rather than say the contents of the workings box). They constructed new meanings which extended their notion of randomness so that the gadget was sometimes random and sometimes not (type Bii). (See incidents 4 and 14 in Table 7.2.) Certainly if randomness is seen as a property of the actual results rather than the process by which those results are generated then it is a logical extension of that way of thinking to regard randomness as something that comes and goes.

Another incident involving the extension of randomness (also type Bii) involved the introduction of degrees of randomness. In the pre-interviews a number of children had used terms like 'more random' and 'fairly random' to compare or describe stochastic phenomena. This approach was also articulated in the Chance-Maker microworld.

When Terry used the spinner, he recognised that 1 was more likely than the other outcomes. He referred to the partial control and predictability of the spinner and Joe talked about it as being "quite random".

> Terry immediately reacts to the spinner's appearance by saying that the 1 is more likely (even though it has landed on a 5). Joe agrees with Terry. They get a long sequence of 1's, even though they try out different strengths. They feel that it lands on 1 too much. Terry says that you can control the spinner a bit because it is most likely to land on 1 because it is a bigger area. "It is fairly random". "You will often be right if you predict a 1."
>
> Joe says it is quite random as it usually lands on 1. "You will usually be right if you predict a 1."
>
> (1.3.1-2)

Under certain circumstances, such as when they were aware of contradiction between their meanings for randomness, they would use the notion of degrees of randomness, but, at other times, they would continue to use their more primitive ideas of unpredictability and fairness. I suggest that a local meaning, which allowed degrees of fairness, was the result of a co-ordination of unpredictability and fairness. Note that this interpretation is quite different from a misconceptions-oriented interpretation in which the boys' use of degrees of randomness might be seen as a new way of thinking in which they have successfully replaced misconceived (or simply wrong) meanings for randomness.

A third meaning is cued instead(Type C)

Contradictions were sometimes neatly side-stepped by the adoption of a different local meaning. Anne and Rebecca had just generated a pie chart for the coin and were perplexed because it appeared to be about 50% heads and tails but they knew that the computer gadget could not be flicked in the way that a real coin could.

> Anne: "I think it might be different because I think we probably control it .... actually, I don't know." Rebecca: "I'm not too sure because .... it might be .... the same or it might not." Anne adds, "A real coin sort of might be fair because you can't really estimate what it's going to come out, and you can't estimate on that really." Rebecca agrees.
>
> (6.7.2)

The contradiction was partially avoided; whereas the control issues for a real coin were different from the computer's gadget, unpredictability was common to both. With a battery of meanings available, it was often possible to find one which seemed to fit the data and this seemed a more urgent need than the resolution of apparently contradictory meanings. Anne and Rebecca were more concerned to find a way of making sense of the coin's behaviour than reflecting on why other meanings did not support the sense-making process.

Different contexts are associated (Type D)

Contradiction was also dealt with on occasions by bringing together two separate domains. For example, Lee and Lenny had just been working with the roll-a-penny gadget when I referred back to the Frisbee, asking them if they thought it was random.

> I remember that I didn't ask them this question in relation to the Frisbee, so I ask now whether they think the distance that the Frisbee goes is random. Lenny: "Yes." Lee: "Yes, because the wind can change."
>
> (5.5.2)

The difference between controlling the Frisbee in everyday circumstances and doing so on the computer was resolved by simple association through metaphor. Whatever it was that had happened on the computer to bring about this behaviour was seen as equivalent to what the wind would do to the everyday Frisbee.

One particularly powerful type of association occurred when the programming code was seen as encapsulating the randomness.

> Anne: "On the computer, it can choose which one it wants. You can't really choose." Rebecca: "You can't decide yourself whether it's going to be heads or tails." I explain again that I am trying to programme the computer so that it behaves as much as possible like a real coin, so that we can't distinguish the results from a real coin. Rebecca: "I suppose if you used the **choose-from**, you could probably choose it by clicking on that and telling it to throw, it would probably throw the one that you chose."
>
> (6.7.7)

Rebecca recognised that there was a sense in which control was in fact exerted by the user through the use of the **choose-from** primitive and that this was analogous perhaps to flicking the coin in everyday situations.

Behaviour is explained in terms of deterministic or mystical forces

A very common strategy for dealing with contradictory meanings, at least temporarily, was to conjecture that perhaps there were deterministic causes that lie behind the unexplained behaviour. Neil and Gurdev were using the spinner.

> They edit the workings to read: **choose-from [1 1 1 2 2 3 3 4]**. The pictogram shows most 3's, then 2's, then 1's and least 4's. They are both surprised. Neil says, "How come?" They wonder if it is because they changed the strength. They put the strength back to 90 and try another new 50 trials. The pictogram shows most 1's and least 4's, rather as expected. I ask, "The first time we did it with strength 90, we had the 2's and the 4's about equal, didn't we? This time we have done it with strength 90, and it's quite a bit more like what we expected, really, with the 4's least, and the 1's most. How do you account for that? What's going on here?" Neil: "Well, I thought a 100, the speed's going faster, I thought it would have been landing on the 1 most of the time. But strength 90, it's more 1's and lower speed."
>
> (8.8.14)

The search for deterministic explanations characterised much of Neil's and Gurdev's activity. Sometimes, the explanations seemed more mystical than deterministic. Neil and Gurdev had been using the coin, and looking for patterns in the results, when this incident occurred.

> I ask, "With a real coin, do you think you would be getting heads most of the time?" Neil agrees. I ask why. Neil: "I don't know. I just think that heads is ...." Gurdev interjects, "More lucky." Neil continues, "Yes." I ask Gurdev if he thinks you get heads most of the time. Gurdev: "Well, I don't know." Neil: "Actually, I don't think so because it all depends on how hard you flick it, But I'd say more would go on heads." Gurdev: "Yes, that's it. My sister thinks that, if it is on tails, it will be on heads, and, if it's on heads, it will be on tails."
>
> (8.7.3)

Neil believed that heads were more likely in everyday coin-tossing and looked for explanations in terms of how the coin was flicked.

These incidents have been picked out as typical of the attempts to deal with apparently contradictory meanings through explanations involving deterministic or mystical forces. I shall discuss such incidents shortly but first allow me to summarise the incidents involving problematical situations.

Table 7.2 below provides a matrix of incidents in which contradictory local meanings for random behaviour were or may have been apparent.

	Apparent Contradiction	Children's Response	Children	Ref.
1.	Spinner is partially predictable	Degrees of randomness (Bii)	Terry & Joe	1.3.1-2
2.	Spinner is partially controllable	Random apart from 1 (Bi)	Terry & Joe	1.8.3
3.	The coin is sometimes predictable and sometimes not	None (A)	Ray & Luke	2.3.2
4.	Prediction sometimes works for the coin	You can control the coin for a little while. Then it does what it wants. (Bii)	Steve & Richard	4.2.2
5.	Spinner is not fair and yet it is unpredictable	Fairness is ignored (A)	Steve & Richard	4.3.2
6.	The Frisbee is controllable in interview but not on the computer.	In everyday Frisbee throwing, the focus is on control. On the computer the focus is on the variation in distances. (C)	Steve & Richard	4.4.1-2
7.	The computer controls yet the Frisbee is unsteerable	Factors like the wind are associated with the unsteerability (D)	Lee & Lenny	5.5.2

8.	The position of the number on the spinner influences the result yet the spinner is fair (according to its appearance)	The position does not matter very much (A)	Lee & Lenny	5.8.17
9.	The computer controls; the strength control does not.	The spinner is not random (A)	Anne & Rebecca	6.3.2
10.	You flick a real coin but you do not flick the coin gadget	In both cases, you can not predict the outcome (D)	Anne & Rebecca	6.7.2
11.	On the computer, it chooses. In everyday life, you can't choose	*Choose-from* mediates where the control lies (D)	Anne & Rebecca	6.7.7
12.	The spinner is unfair but it is also unpredictable and perhaps unsteerable	Spinner is not random (A)	Cathy & Lynn	7.3.1
13.	The Frisbee is not totally unpredictable and not totally unsteerable	Lynn: it is random Cathy: it is not random (A)	Cathy & Lynn	7.4.2
14.	Sometimes there are patterns (in the coin's results) and sometimes there are not	You lose concentration and forget what the pattern was (Bii)	Neil & Gurdev	8.2.5
15.	Patterns are sometimes evident in results of coin	Sometimes the coin is random and sometimes it is not (Bii)	Neil & Gurdev	8.2.6

16.	More tails than heads contradicts Neil's view that there would be more heads with a real coin	It depends on how hard you flick it (E)	Neil & Gurdev	8.7.3
17.	Few 1's after 50 results with spinner (set to contain several 1's)	When the speed is faster, it lands on 1 more often (E)	Neil & Gurdev	8.8.14

Table 7.2 : The occurrence of incidents involving contradictory local meanings

The table does not list all such occasions but has identified those which can be regarded as relatively clear examples of the strategies adopted.

In fact, the search for deterministic causes of short term behaviour drove much of the activity even when the process was more to do with general sense-making than dealing with specific contradictory meanings. The next section considers their meanings for deterministic behaviour and the relationship with meanings for stochastic behaviour.

7.4.9. Expressions of Meanings for the Deterministic and the Stochastic

On many occasions, the short term behaviour of the gadgets was explained by reference to cause and effect. There was a strong tendency to search for deterministic type explanations in their attempts to construct meaning for that behaviour.

A common occurrence, for example, was to explain the behaviour of the spinner in terms of the position of the numbers on the spinner. Terry and Joe tried but failed to explain the lack of 2's in their results in terms of the position of the 2 on the spinner.

> After 24 results, I suggest that they do a pie chart. Terry predicts that the pie will look like the actual spinner. He notices that there is no 2. He says that he might have expected a lack of a 2 to go along with a lack of a 5 because they are next to each other on the spinner, but they have in fact had a 5. He tries to explain this phenomenon in terms of which way the spinner is turning round but fails to do this.
>
> (1.8.2)

After some further experimentation, Terry and Joe found that 2's did begin to appear in the results and they concluded that the spinner was random (apart from 1 for reasons of fairness — 1.8.3).

A typical way of working was for the children to begin by conjecturing possible deterministic reasons for a gadget's initial behaviour. Further results would not support that conjecture and an alternative deterministic conjecture would be found. When this failed too, perhaps yet another reason for the behaviour would be proposed though at some point, if sufficient testing were carried out, the search for such explanations would come to an end and the behaviour would be described as random.

Consider this example. Ray and Luke were also using the spinner when they noted that "it always lands on 1" (2.3.1). At this stage Ray conjectured that you could make the spinner land on any number that you wanted, perhaps just by giving the arm a slight push. After a few more trials though, Ray and Luke found other outcomes appearing, prompting Ray to declare, "It's quite unpredictable...It depends where it actually starts off" (2.3.2).

At this stage, the boys were giving a deterministic explanation for stochastic behaviour, in the sense that factors, which were out of their control, were giving rise to the unpredictability. In fact, I pointed out that the arm always began at the top of the spinner. Ray and Luke then conjectured that the spinner was predictable for 100% strength but unpredictable for other outcomes. They tried some low strengths and found that the spinner was indeed unpredictable. They remembered however that at the beginning, using 100% strengths, they had experienced a run of 1's.

Another attempt at 100% strength persuaded them to change their views again.

> They try 100% as Ray makes this assertion only to find it land on 5. This surprises them. Ray jokes that Luke stopped it by touching it. Then they get a 3 and then a 4, all using 100% strength. Ray says, "It's unpredictable, very unpredictable." Luke says, "It's unpredictable, if you like, it depends, when you hit it at strength, it depends you know the way that the spinner goes. It depends how big the numbers are as well... because on the 1, it's half the size of the circle, but on the rest, it's just really small."
>
> (2.3.2)

The unpredictability was explained in terms of a lack of control over factors which they believed would influence the outcome, including cases where the strength was 100%. The influence of the size of the sectors began to emerge as an important factor.

The size of the strength, not surprisingly, was very frequently seen as a cause of the outcomes. When using the coin, Ray and Luke began by conjecturing that strength determined the outcome of a head or a tail.

> They begin to use the coin. Almost immediately Ray observes that he thought the outcome was going to be the same. Ray explains that this was because he flipped the coin with 100% both times. He tries with 100% again. He gets a head again and says, "It does come out the same." ..... Ray says, "50% is always different." Luke agrees. In fact, the next 50% does come out as a tail, confirming their thinking. Again they get a head. Ray says, "100% is like smashed down, stays, doesn't bounce or anything." Luke continues, "50% is always different." Ray says, "So it is quite predictable." ....Then they get another tail, instead of a head as their method predicts. They note that it is different but are quite bemused by this. They try again and get another tail. Ray tries 100% instead and gets a head, instead of a tail (by their predictive method). Ray says, "These are random."
>
> (2.2.1-2)

The notion of randomness emerged out of a failure to find consistent explanations in terms of the strength. This pattern of behaviour was common throughout many of the eight case studies. A second example can be found in Steve and Richard's early use of the coin gadget (4.2.1-3). Patterns of results seemed to be related to the strengths used. At one point, Steve and Richard felt that the coin was steerable at low strengths but not high, an idea which probably matched their everyday experience. Eventually though this and other conjectures were abandoned in the light of experience.

> Steve says, "No, you can't control it at all." Richard agrees saying, "It's random .... it can land on any side." I ask, "If there was a pattern to it, would you feel it was random?" They both say, "No."
>
> (4.2.3)

Further examples of the use of deterministic explanations are listed in Table 7.3.

Prompt	Explanation	Children	Reference
Lack of 2's on the spinner	2's and 5's next to each other. But no lack of 5's. Tries to explain in terms of direction of spinning.	Terry & Joe	1.8.2
First three tosses of coin, all at 100%, are heads, then 50% gives a tail	It comes out the same because 100% smashes the coin down but 50% strength is different.	Ray & Luke	2.2.1

First few spins of spinner all result in 1.	It depends on where the spinner starts.	Ray & Luke	2.3.2
First clicks on the spinner, each at 40%, give the same results	Same strength gives about the same result. Unsteerability is later explained in terms of different surfaces and because it is the computer that controls the outcome.	Donna & Rose	3.3.1
Alternating results in tosses of coin	The strength determines the result.	Steve & Richard	4.2.1
Some successful prediction of the coin	Small strengths give predictable results	Steve and Richard	4.2.2
The spinner is 1 most of the time but not quite always	It depends how hard you spin it. It goes on 1 when you do small strengths	Steve and Richard	4.3.1-2
Results on the spinner include many 1's.	Slower spins seem to land on the 1	Anne & Rebecca	6.3.2
The Frisbee goes to 100 several times	A strength of 100% causes a result of 100.	Anne & Rebecca	6.4.1
The Frisbee (and the coin) land on different results even when the strength is the same	In real life, the Frisbee is affected by the wind (and the coin by the way you spin it).	Neil & Gurdev	8.7.5

Table 7.3 : Occurrences of deterministic explanations

These incidents are picked out from the case studies as relatively clear examples of two processes involving deterministic explanations.

In the first, deterministic reasons are conjectured and then rejected in the light of experience. After several such attempts to explain the behaviour deterministically, a stochastic description will often emerge out of the failure to find any cause which is consistent with further outcomes. Alternatively, the deterministic explanations are not so much rejected as recruited as a sort of explanation of the stochastic. Often the conjectured causes are outside of the control of the child in any case, and so the unpredictable behaviour can be explained in terms of lack of control over those causes.

There is a strong sense in which stochastic phenomena only exist to deal with those aspects of the world which can not be deterministically explained. It is no coincidence that the children's actions usually involve searching for deterministic behaviour since it is impossible to look for stochastic behaviour when it is characterised only in terms of the absence of various attributes: a *lack* of predictability, a *lack* of sequential patterns, and a *lack* of control. A richer appreciation of the stochastic must involve the identification of positive features to which the deterministic can not lay claim and these lie in their long term behaviour to which I turn my attention in Chapter Eight.

7.5. SUMMARY AND DISCUSSION

This chapter has considered the children's expressions of local meanings. The dominant method of constructing meaning for the behaviour of the gadgets was to conjecture deterministic causes for the results and then to test these ideas out through further trials. Experience would typically lead the children to a rejection of deterministic explanations in favour of stochastic descriptions or the causal factors would be seen as out of their control and be recruited to *explain* the random behaviour.

The Chance-Maker microworld enabled the observation of five different meanings for the behaviour of the gadgets:

(i) unpredictability,

(ii) unsteerability,

(iii) irregularity of results,

(iv) fairness, and

(v) computer-in-control.

The fact that the computer was seen in control was sometimes used as a reason for rejecting the notion that the gadgets were random, but, like other deterministic causes, it was sometimes used to justify stochastic behaviour since, from the child's perspective, the computer's mechanisms lay outside his or her control.

Each of these five local meanings was expressed through the tools and structures of the Chance-Maker microworld. Details of how this process was played out in practice have been discussed but involve the webbing of local meanings with specific features of the microworld, such as: familiarity, click-and-play, replication, the strength control, reporting of results (such as in the results box), unfairness (in the gadget's appearance, the workings box, and the results), mending, and the **choose-from** primitive.

The five types of meaning were often associated with one another, almost interchangeably. As a group, these local meanings were typically seen as self-explanatory, except in those cases where the deterministic was recruited as a sort of explaining device for properties such as unsteerability or unpredictability. In other words, each meaning could be used in support of another but there was no other knowledge which was causally connected with this group, other than the absence of causal factors.

Equally there was little evidence of these meanings having consequences other than those expressed by the meanings themselves. Unsteerability can cause unpredictability which can cause irregularity et cetera et cetera.

Even though they were often used in association, it was still apparent that the five meanings could be contradictory, though the inconsistency was not always apparent to the children. The fact that the children were frequently unaware of these contradictions suggests that these meanings are only loosely connected, and that they will be cued by surface features of the context situation. The methods of dealing with such contradictions when they were apparent also supports the notion of weakly connected pieces of knowledge. For example, consider the cueing of a third meaning, which happens to support both sides of the apparent contradiction, rather than an explicit attempt to resolve the contradiction by the replacement of these meanings by a new meaning. If two meanings are strongly connected, one might suppose that a challenge to one would be seen as an equal challenge to the other. The fact that a meaning can be cued to deal with the situation unproblematically suggests that this meaning is not that tightly connected to the original contradictory meanings.

The five local meanings for the behaviour of the gadgets in the Chance-Maker microworld had in many cases equivalent articulations in the pre-interviews (computer-in-control was the exception for obvious reasons). Given this fact and the early emergence during the clinical interviews of these local meanings, it is reasonable to suppose that the structures of the Chance-Maker microworld acted as a window on meanings that had already been abstracted from everyday experience. There is evidence to suggest that in some cases a limited amount of co-ordination had taken place prior to experience with the Chance-Maker microworld. As well as the interchangeability of meanings for randomness in the pre-interview and more clearly in the early work with the Chance-Maker microworld, I note also examples such as Terry's and Joe's use of degrees of randomness, which was apparent in both the pre-interview and the use of the Chance-Maker microworld.

In contrast, the pre-interviews provided little evidence of global meanings, and so the process by which the children in the eight case studies constructed meanings for long-term behaviour of the gadgets is of special importance. I examine this process in Chapter Eight.

CHAPTER EIGHT

The Construction of Global Meanings *In* a Domain of Stochastic Abstraction

8.1. INTRODUCTION

This chapter is the second in a sequence of three which examine meanings for stochastic behaviour during tool use in Iteration 3. As already explained, I refer to the children's meanings in the domain of abstraction so as to distinguish the construction of these meanings from the other closely related strand of this study, which considers the meanings that can be constructed *for* the microworld itself.

The overview for Chapter Seven encompassed this chapter, including the procedures involved in the children's use of the tools in Iteration 3. We have seen in Chapter Seven how the children in the eight case studies of Iteration 3 were able to articulate five separable local meanings for the behaviour of stochastic phenomena. We also observed how these meanings were weakly structured as a group, with one meaning often being substituted by another. These meanings were often regarded as self-explanatory and were rarely discussed in terms of their implications for other pieces of knowledge outside of this closed set of meanings. The only exception was when a deterministic cause of the behaviour, such as when the computer-in-control meaning was recruited to 'explain' short term behaviour; when the computer is in control, the child is not, and so the behaviour appears to be stochastic from the child's point of view.

In this chapter, we examine the children's construction of global meanings — how they came to view as important an increase in the number of trials. The story of how the meaning-making unfolds is one of continuity, and so is best told, at least initially, by a close look at one pair of children's interactions with the Chance-Maker microworld. This case will outline some of the main issues which appear in many of the case studies. Two other cases will be looked at in less detail to support, and in some instances highlight variations from, the main themes raised by the first case. The remaining cases are not presented for the sake of brevity; analysis of all the cases suggests that the significant issues are represented in the three cases below. Detailed analysis of all eight cases can be found at the world wide web address: http://www.warwick.ac.uk/wie/staff/DP.htm

8.2. ANNE AND REBECCA : A CASE OF GLOBAL MEANING-MAKING FOR LONG TERM BEHAVIOUR

I will begin the story of Anne and Rebecca, age 10 years, by consideration of their meanings for long term behaviour as apparent in their pre-interviews and in the early interactions with the Chance-Maker microworld. This will give us a base-line from which the story can unfold.

8.2.1. Initial Meanings for Long Term Behaviour

In the pre-interview, Anne recognised that on a normal dice no number was easier or harder to obtain than any other number , a view which seemed to be linked to the fairness of the dice. When I asked her how she might check to see if a dice was unfair, she replied that she would roll the dice, but she seemed to believe that rolling the dice three times would be sufficient "because it would probably come up the same number every time". She offered no suggestions as to what else she would do. Unfairness was identified through a short sequence of throws of the dice.

Rebecca also argued that the numbers on a dice were equally easy or hard to obtain but her explanation was in terms of the dice's randomness. Like Anne, Rebecca suggested rolling would uncover an unfair dice. She suggested that unfairness might manifest itself in too many occurrences of one number: "You could roll it a few times, maybe ten or twenty times and if you get the same number, or one number more times than others." Rebecca did seem to have some sense of fairness, the numbers should come out equally often, but she did not discriminate between short term and long term stochastic behaviour. For her, the fairness meanings applied in either context.

These impressions from the pre-interviews were supported in the early interactions with the Chance-Maker microworld. In the following extract, the two girls had already used the various gadgets at top level. I had recently shown them the tools inside the coin gadget. Anne and Rebecca had spent a little time playing with the coin gadget, trying out 35 trials in all.

> After 35 trials they look at the results box. Rebecca asks whether there is a limit to how many times you can toss the coin. I say that there is no real limit as the limit is very large, and ask, "If you are trying to decide whether the coin is working properly or not, what do you think a good number of times to throw it would be?" Rebecca says, "About ten." I ask, "Do you think there would be any advantage in doing it more than ten?" Rebecca: "Yes because you get like a clearer answer." I ask how many times would be better than ten. Rebecca: "Twenty." Anne agrees. I ask again if more than 20 would be advantageous. Rebecca says, "Yes." Anne says, "I don't think so."
>
> (6.7.3)

My interpretation is that Anne, who was thinking very much in terms of the sequence of results, was unable to see any advantage in doing more than 20 trials, since 20 was quite sufficient to identify recurrences of the same number. Rebecca in contrast did see the need for more trials in order to identify if one number appeared more often but believed that the fairness in numbers of outcomes could be picked up in only ten or twenty trials, though more might be better.

Below, I outline their subsequent interactions with the coin, spinner and dice gadgets.

8.2.2. Outline of Anne's and Rebecca's Use of the Coin, Spinner and Dice Gadgets

Anne and Rebecca use the coin gadget

There was some dissatisfaction that so many tails were being produced, and so I asked again (6.7.8) if there was any advantage in doing more tosses. This time they both agreed that there was (6.7.9).

> This time they repeat 100 new trials of the coin gadget. The pictogram shows more tails (Figure 8.1). Anne says, "I think the tails is more popular .... It seemed like first there more heads were coming up, but now loads more tails are coming up." I ask if they would get the same picture if they did it again. Anne: "Yes." Rebecca: "I'm not sure. It might be 50 / 50; it might be more heads." Anne: "I think it will be more tails."
>
> (6.7.9)

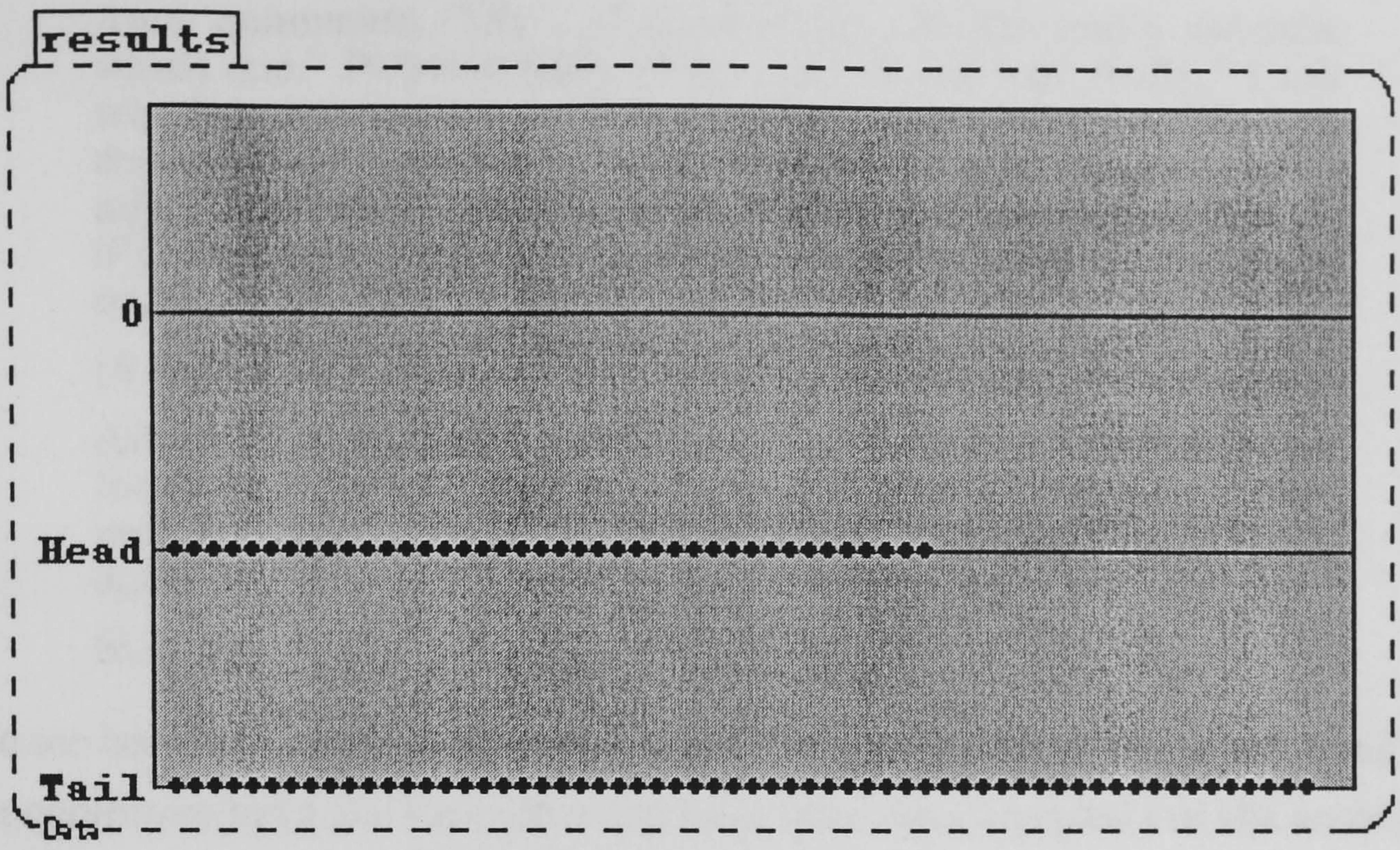

Fig. 8.1 : The pictogram shows more tails

Anne decided, it seems on the basis that more tails had appeared in both the experiment with 100 trials and that with 50 trials, that more tails was a tendency of the gadget, and would be repeated in another experiment. Rebecca was less inclined to trust in the results so far, and believed it might still come out about equal. When they repeated a new set of 100 trials, there were more heads (Figure 8.2). That was sufficient evidence to persuade Anne that she had been wrong.

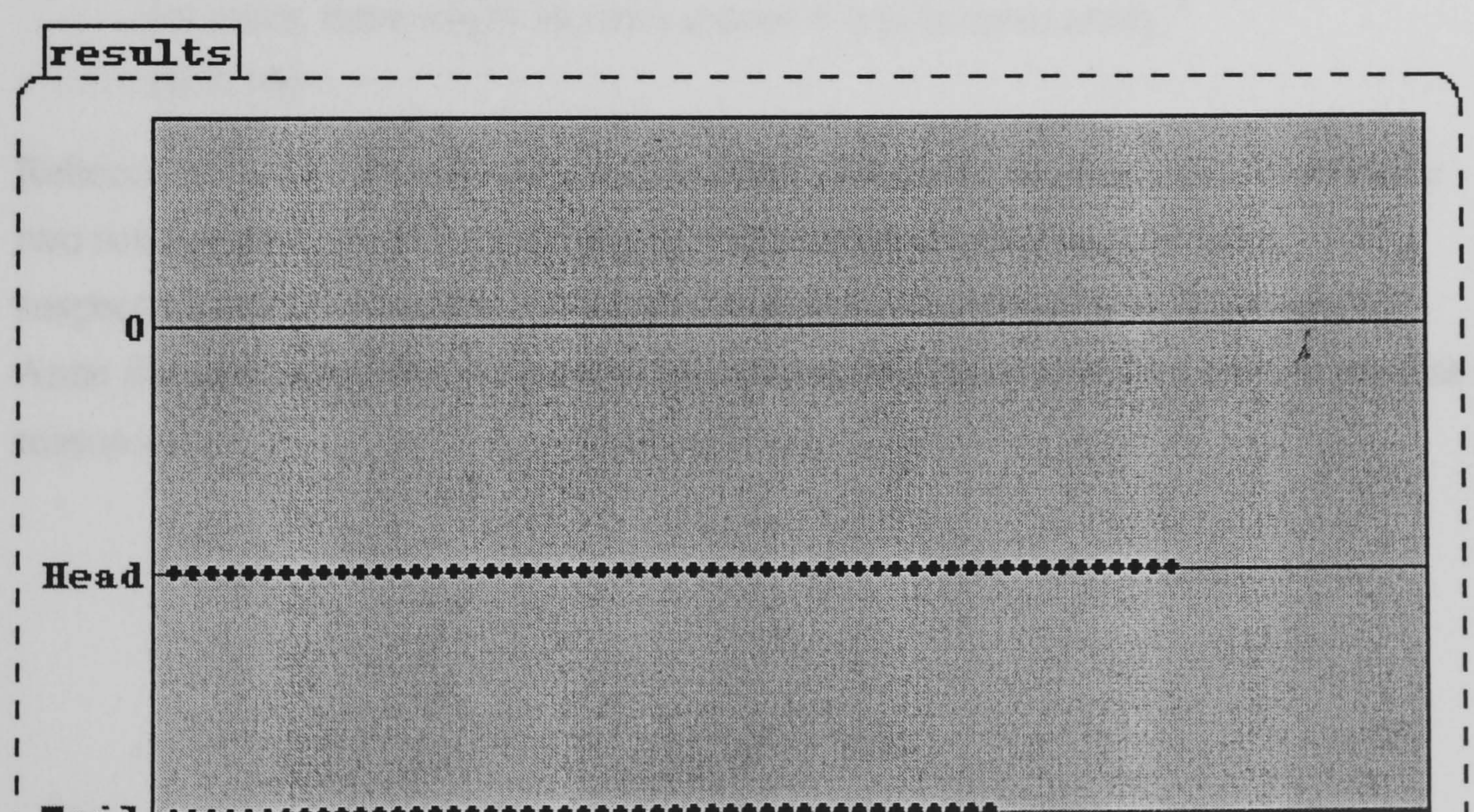

Fig. 8.2 : When they repeated a new set of 100 trials, there were more heads

> Anne comments, "Oh .... I don't think you can really estimate which one." Rebecca adds, "You can't be too sure really." I ask whether you could be sure with a real coin. Rebecca: "Well, it depends what side you start on, I think ....If you start on tails, it might land on tails again because it might not be a very good flick. If it's heads, and you flick it, and it isn't a very good flick, it will land on heads."
>
> (6.7.9)
>
> After looking at the pie chart, Anne says, "The first time there were loads of tails, so I thought it was going to be tails again. But probably after a couple of goes, it will probably do tons of heads again."
>
> (6.7.10)

We see how the facility of being able to repeat experiments enabled Anne to test out her conjecture that tails were appearing more often. She accepted that she could not predict whether heads or tails would appear more often but now conjectured that there was some type of oscillating pattern. Rebecca was not prepared to extend the domain of her abstraction (that you can not be sure) since in the everyday world the outcome was influenced, she claimed, by how you flicked the coin. I probed what they thought would happen if 200 trials were repeated.

Anne replied:

> "Probably it will be about even, I think .... because, if it is quite a lot more, there might be more chance it will be even really."
>
> (6.7.10)

Rebecca said that she was not too sure. After 200 trials, the pictogram showed the two rows nearly equal in length (Figure 8.3). Rebecca was unconvinced, suspecting that the evenness may have been just coincidence (6.7.10), whereas Anne thought there was a reason for this pattern but she was unsure as to what that reason might be.

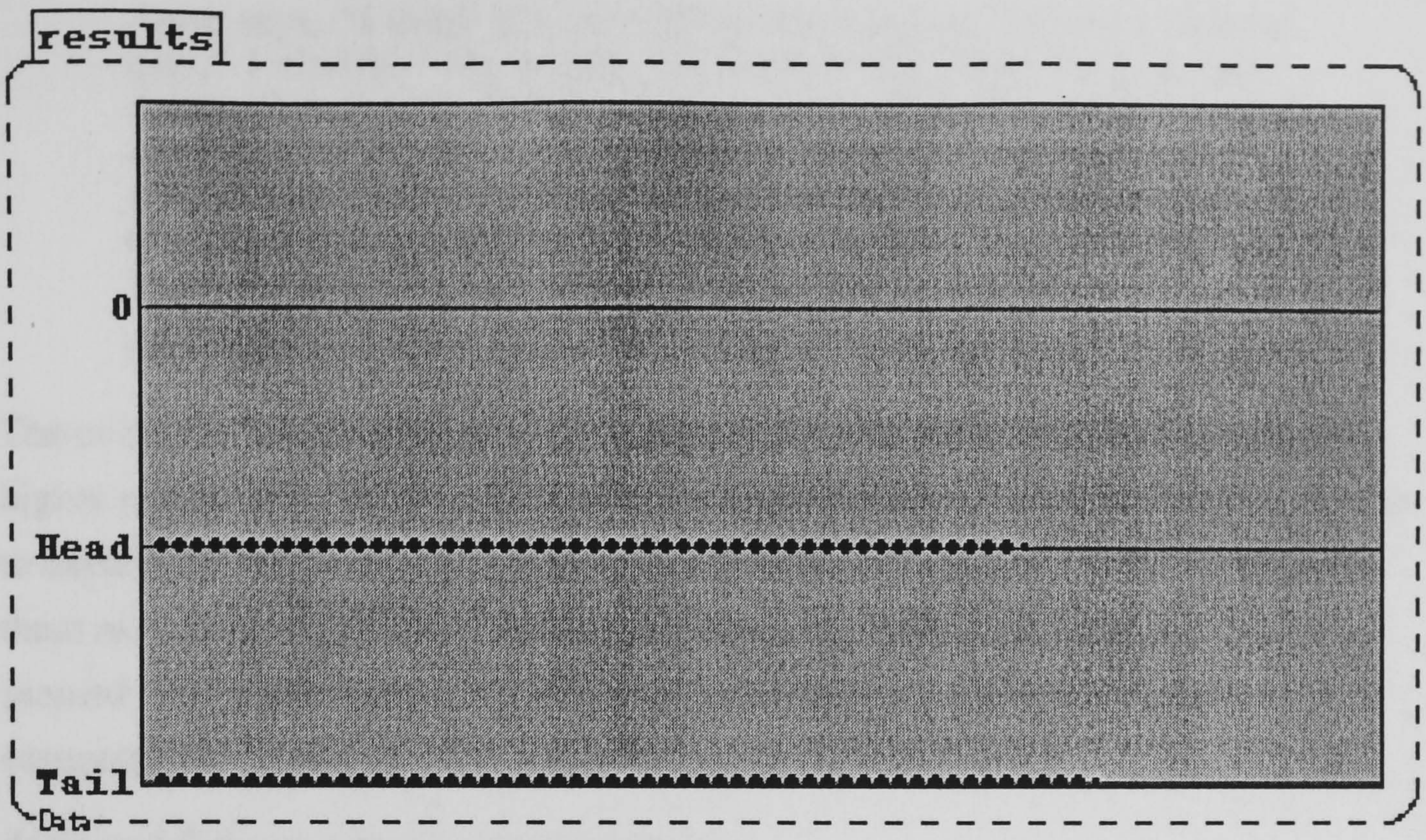

Fig. 8.3 : After 200 trials, the pictogram showed the two rows nearly equal in length

Anne had conjectured and rejected a series of causal explanations for the erratic behaviour when the number of trials had been too large in her view to dismiss out of hand. In fact, the number of trials was not large enough for the behaviour to settle down into predictable patterns. This between number of trials proved to be a problematic region for many children, and I came to refer to a 'large small' number of trials. Eventually Anne had constructed a situated abstraction that larger numbers of trials will generate even results. Rebecca was not yet convinced of Anne's ideas though she did agree that it would perhaps be even when they carried out 500 trials (6.7.11).

They accidentally did 1000 trials. They were now both predicting that the pie chart would be even. They looked at the pie chart (6.7.12) which was indeed almost even (Figure 8.4).

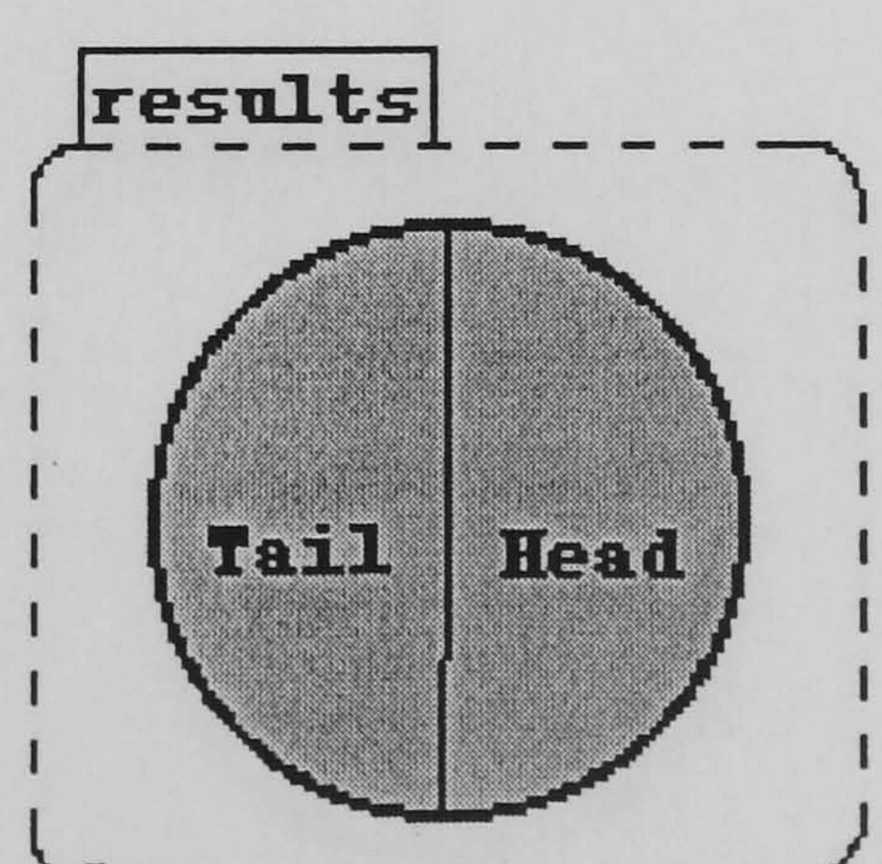

Fig. 8.4 : They looked at the pie chart which was indeed almost even

Anne says, "I think it's the highest the number, the even more it gets." I clarify, "The higher the number, the more even it gets." Anne: "Yes." I ask Rebecca if she agrees. Rebecca: "Because the other time, when we did less numbers, it was half um even really." I ask, "Do you agree with that — the more times you do it, the more even it's getting?" Rebecca: "Yes, it seems to be." I ask if that would be true of a real coin. They both think it would.

(6.7.12)

The evidence had encouraged them to construct the situated abstraction that the higher number of trials, the more even the pie chart became. Rebecca was now able to explain the earlier results where there had been less consistency by thinking of them as 'half-even'. They were so confident of this situated abstraction that they seemed to be prepared to extend the domain of abstraction to real coins in non-computational contexts.

Anne and Rebecca use the spinner gadget

After some initial work with the spinner (6.8.1 to 6.8.2), Anne and Rebecca edited the workings to make the spinner fair (6.8.3) (Figure 8.5). The workings now read: **choose-from [1 2 3 4 5]**.

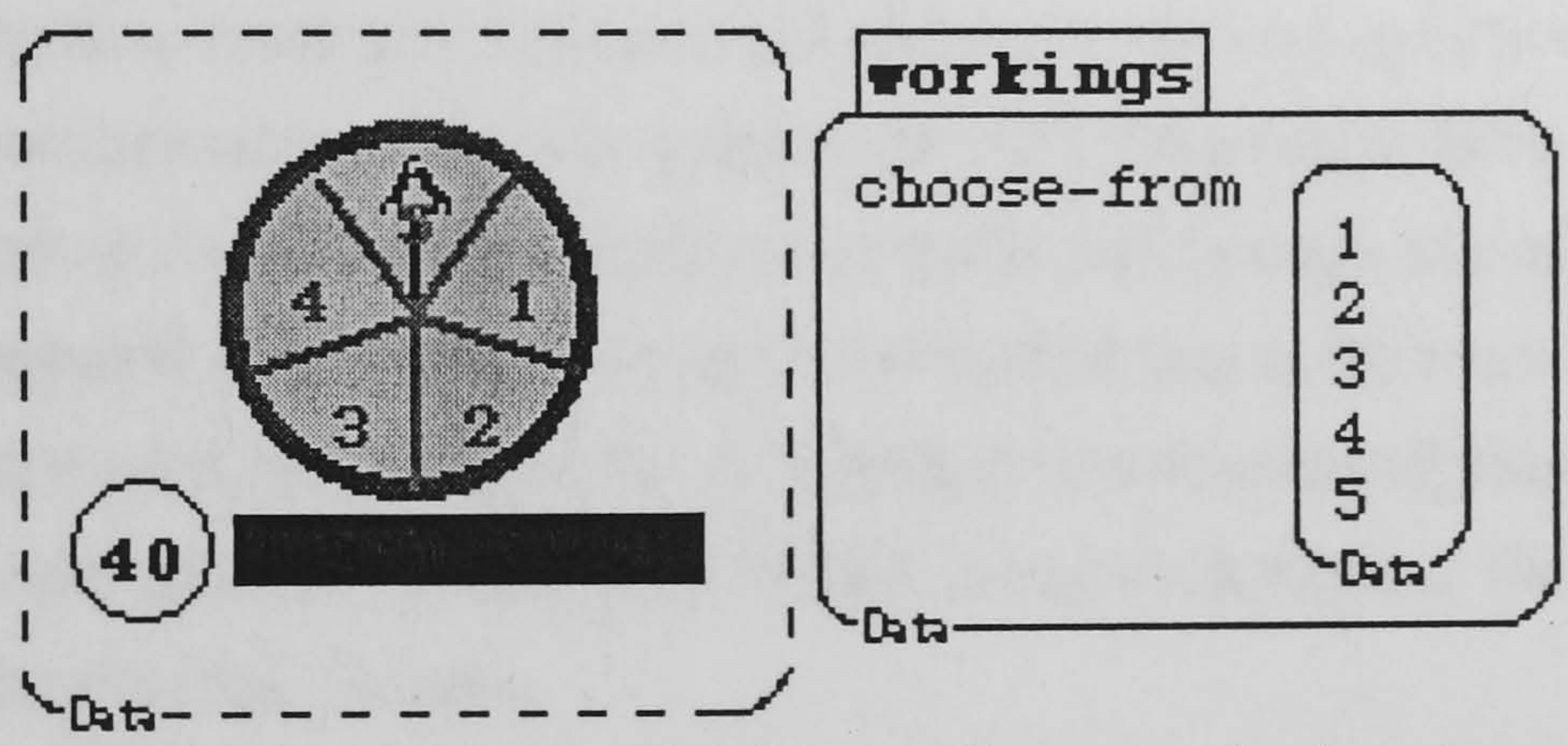

Fig. 8.5 : Anne and Rebecca edited the workings to make the spinner fair

They repeated 50 trials and found that the 5's and 2's came out most often with 1's the least (Figure 8.6).

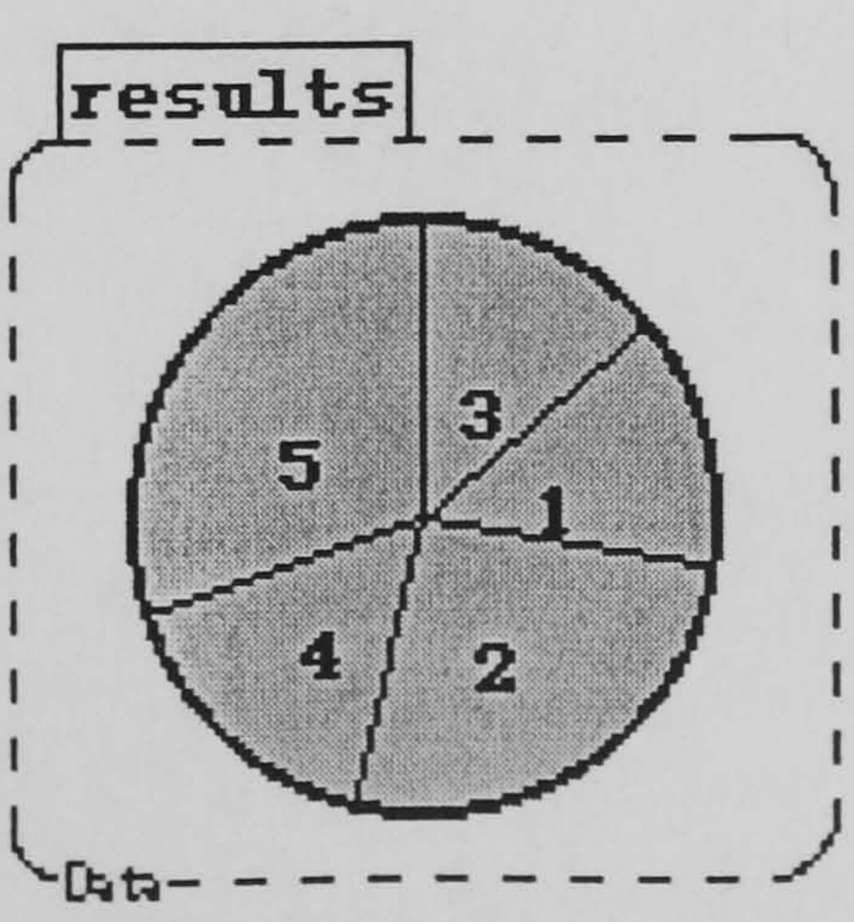

Fig. 8.6 : The 5's and 2's came out most often with 1's the least

I ask, "If your aim was to make that pie chart look more even, what would you do?" Anne says, "I'd make the 5 a bit smaller ...." Rebecca interjects, "I'd make the others a bit bigger." They begin to edit the workings. Anne says, "Why don't you put them all the same number (Figure 8.7)? That would be even then. Like put three on 1, three on 2. That would be fair because that would be even then." In fact they edit the workings to read: **choose from [1 1 2 2 3 3 4 4 5 5]**.

(6.8.4)

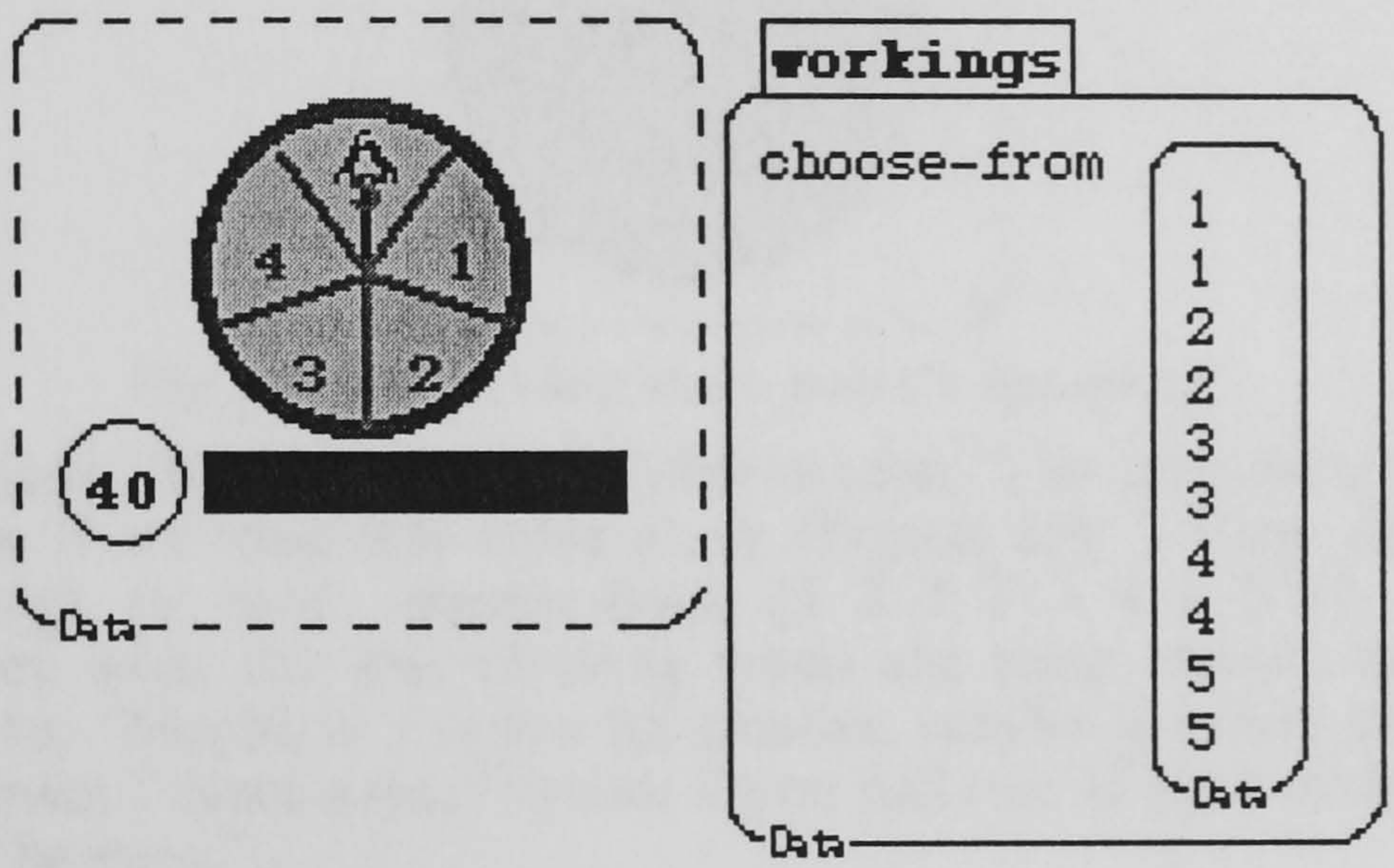

Fig. 8.7 : Why don't you put them all the same number

It is interesting that Anne and Rebecca did not respond to my question by suggesting that they repeat a higher number of trials. One might have expected this response based on the situated abstraction constructed through use of the coin. They were prepared to extend the situated abstraction that more trials would generate a more even pie chart for the dice gadget to non-computational dice, but not to the spinner gadget. Instead they looked towards changes in the workings to bring about the desired changes.

Surprised that they did not seem to see the connection between the spinner's and the dice's workings, I asked what the workings were saying.

Anne: "Well, I think it is going to be more even now. You can't really estimate what it's going to be on. Much more even." They change the strength to 10 and do 50 new trials. The pie chart shows most 1's and least 4's (Figure 8.8).

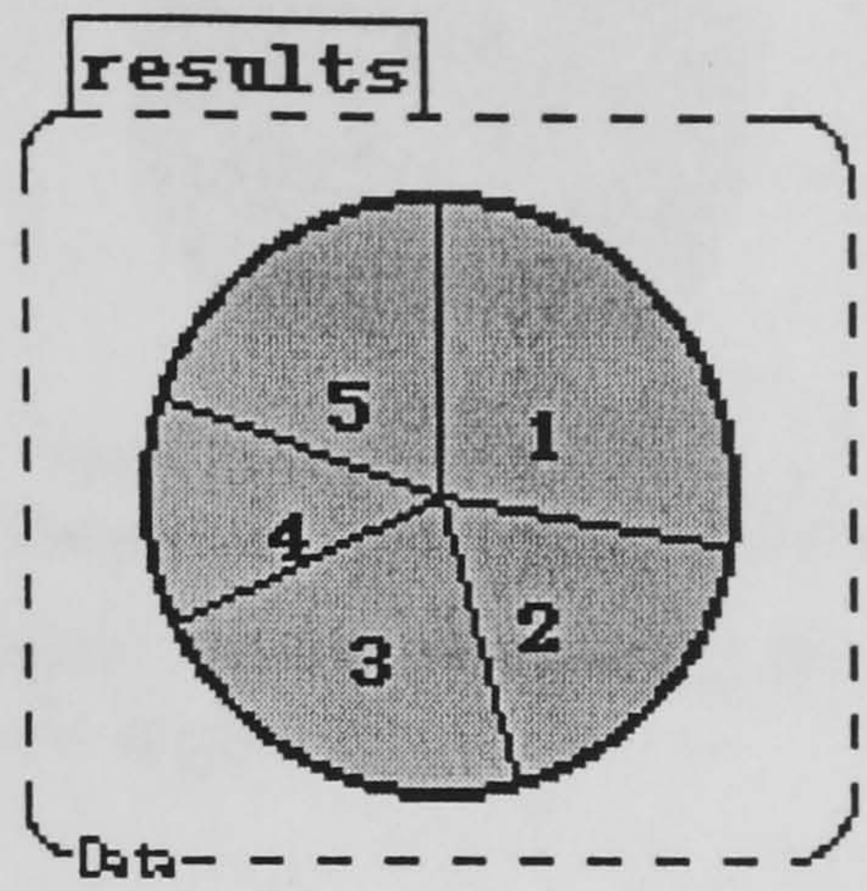

Fig. 8.8 : The pie chart shows most 1's and least 4's

Anne says, "More 1's again." Rebecca says, "I wonder what would happen if we took one more away (Figure 8.9)." They edit the workings to read: **choose-from [1 2 2 3 3 4 4 5 5]**. I ask Rebecca what she was thinking when she made those changes. Rebecca: "Maybe if 1 was a bit smaller, maybe it would be a bit more even." Anne says, "I think if you had one of each number, it would be even."

(6.8.5)

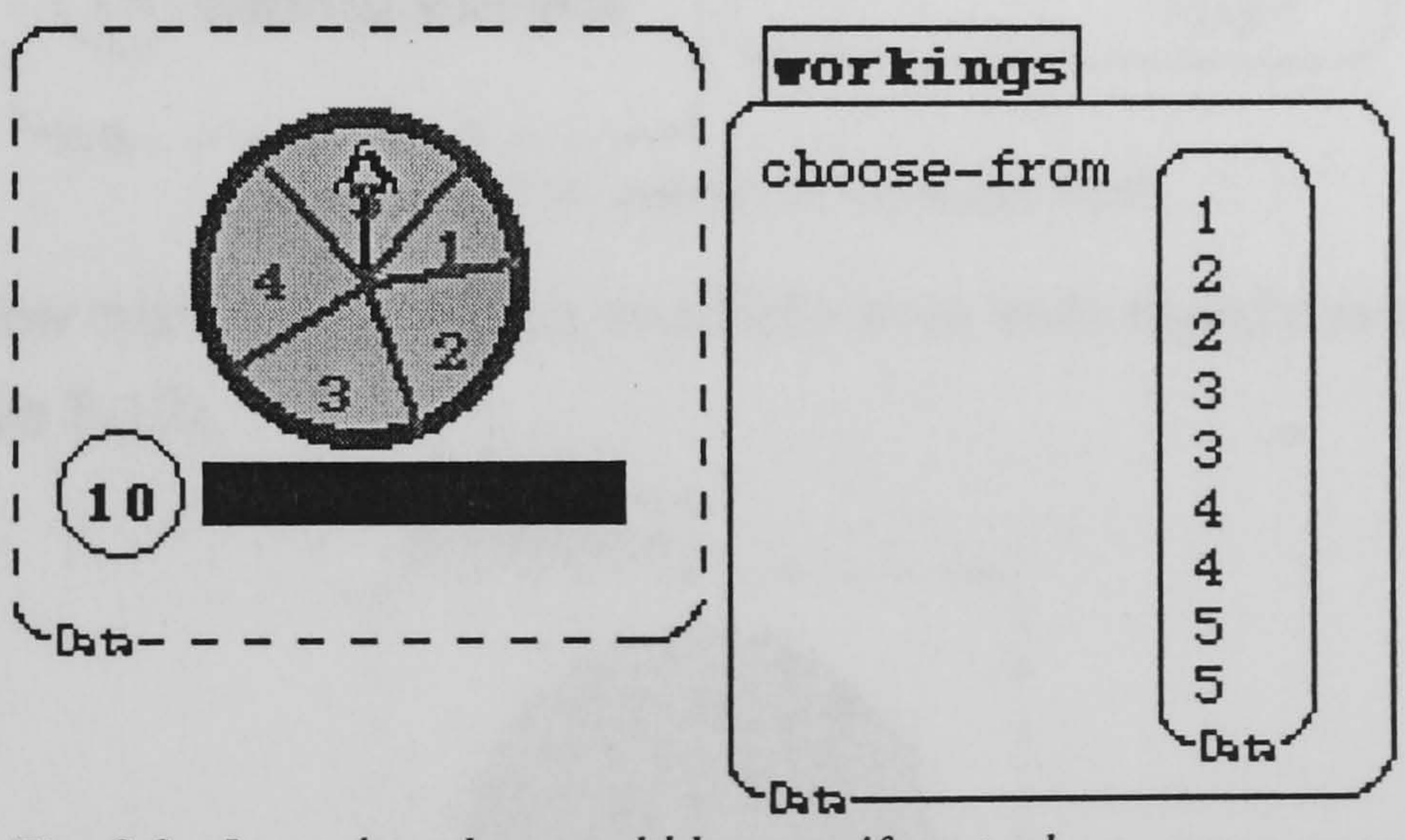

Fig. 8.9 : I wonder what would happen if we took one more away

Rebecca was responding in a quasi-deterministic fashion, believing that changes in the workings would have direct consequence on the results, even though the number of trials was not large. They repeated 50 new trials. The pie chart showed most 3's and least 2's (Figure 8.10).

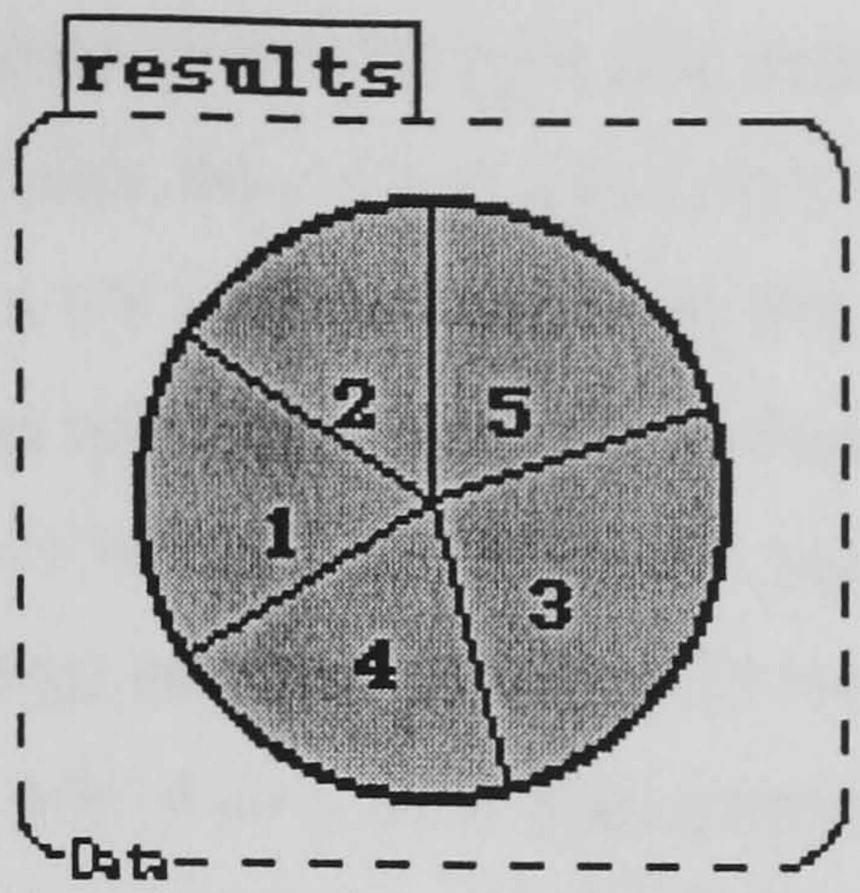

Fig. 8.10 : The pie chart showed most 3's and least 2's

> Rebecca says, "Maybe if that *(pointing to the 1 sector on the spinner)* were a tiny bit bigger."
>
> (6.8.5)

They edited the workings back to read **choose-from [1 2 3 4 5]** (Figure 8.11).

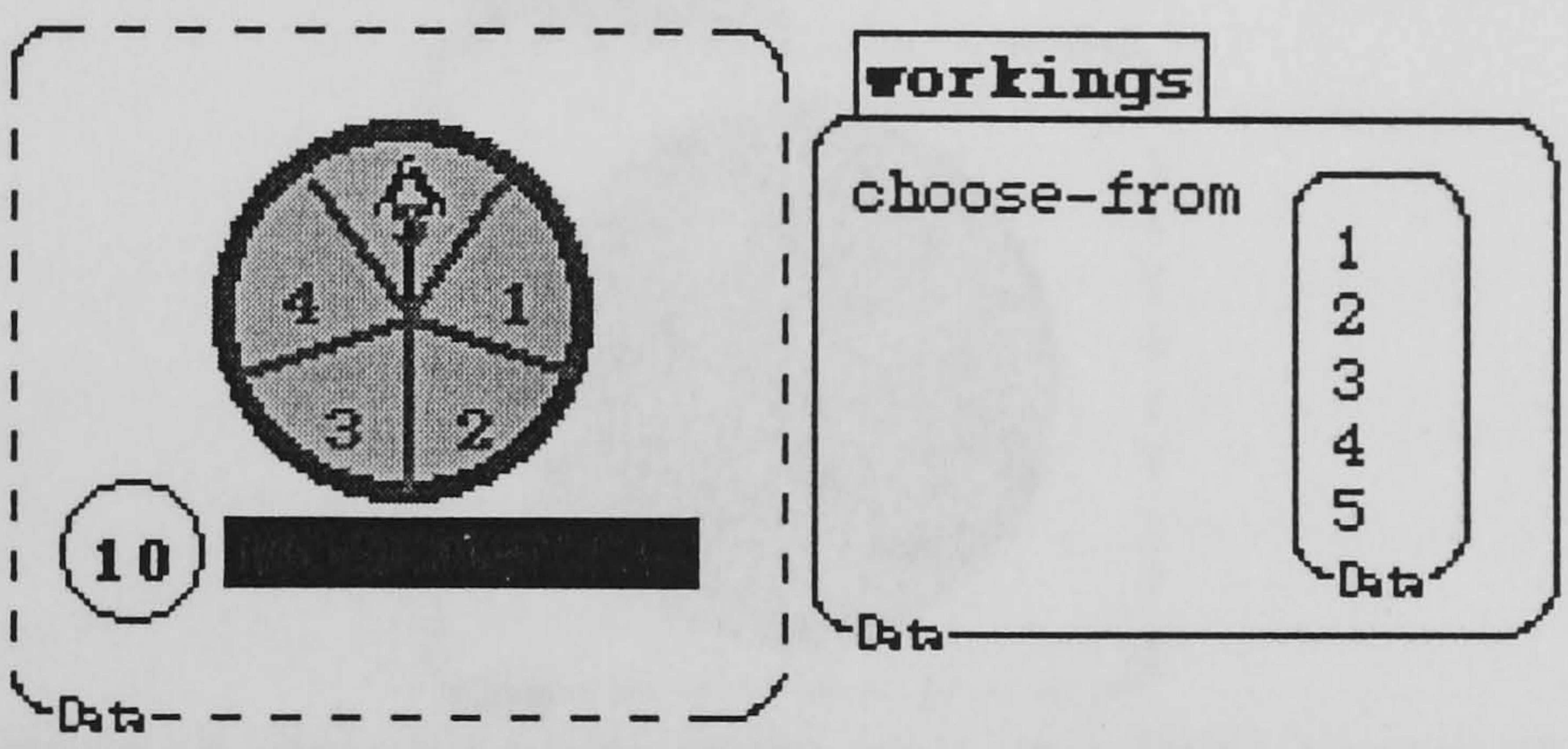

Fig. 8.11 : They edited the workings back

They did 50 new trials. The pie chart was fairly even with slightly more 1's and less 2's (Figure 8.12).

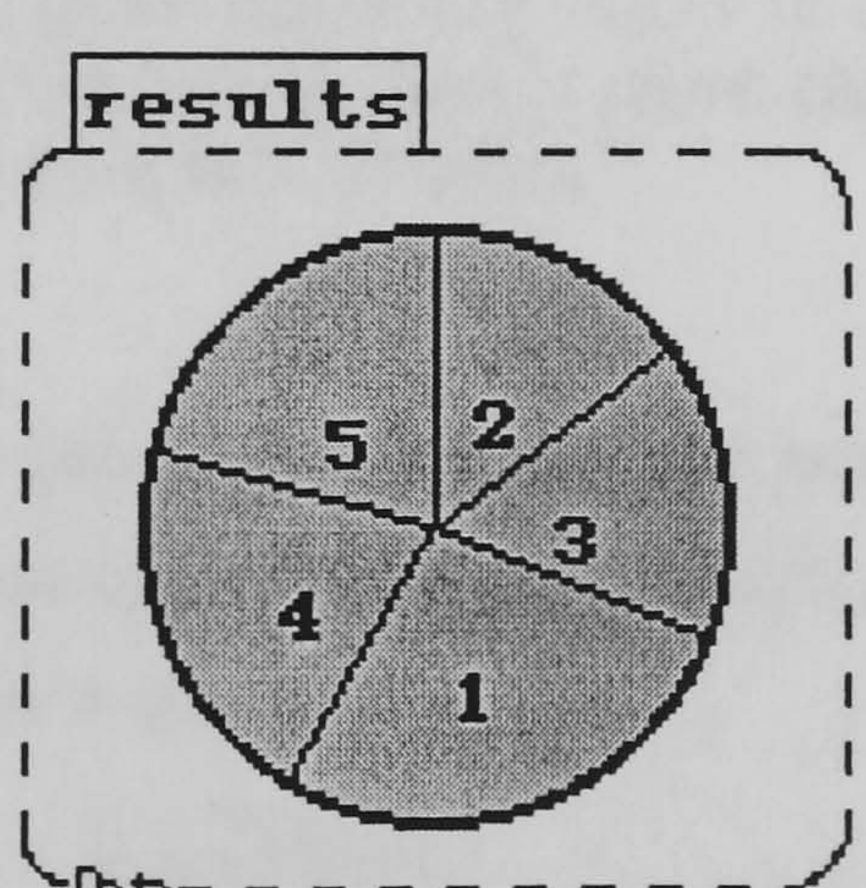

Fig. 8.12 : The pie chart was fairly even with slightly more 1's and less 2's

> Anne says, "It's a tiny bit even but there are more 1's." The pictogram confirms this. Rebecca: "Yes, definitely more 1's." Anne: "Yes, but it's fairly even, if you see what I mean." I ask, "So, if you are trying to make this pie chart more even what could you do?" Rebecca: "Maybe throw it more times like we did with the coin." Anne says, "Yes."
>
> (6.8.6)

Rebecca has moved away from her previous quasi-deterministic strategy in favour of the situated abstraction drawn from the coin that more throws leads to more even pie charts. It is interesting to note that the connection between the coin and the spinner was far from obvious for Rebecca. She needed considerable experience with the spinner, forming and rejecting conjectures based on intuitions drawn from her deterministic view of the world, before she was prepared to consider her recently acquired coin-based situated abstraction. In view of the preceding activity, it is quite possible that there was now a tacit recognition that the workings needed to be fair – a sort of condition on the domain of applicability of the situated abstraction.

They decided to try 150 trials (6.8.7). The pie chart for 150 trials was more even with slightly more 4's and 3's, and less 2's (Figure 8.13).

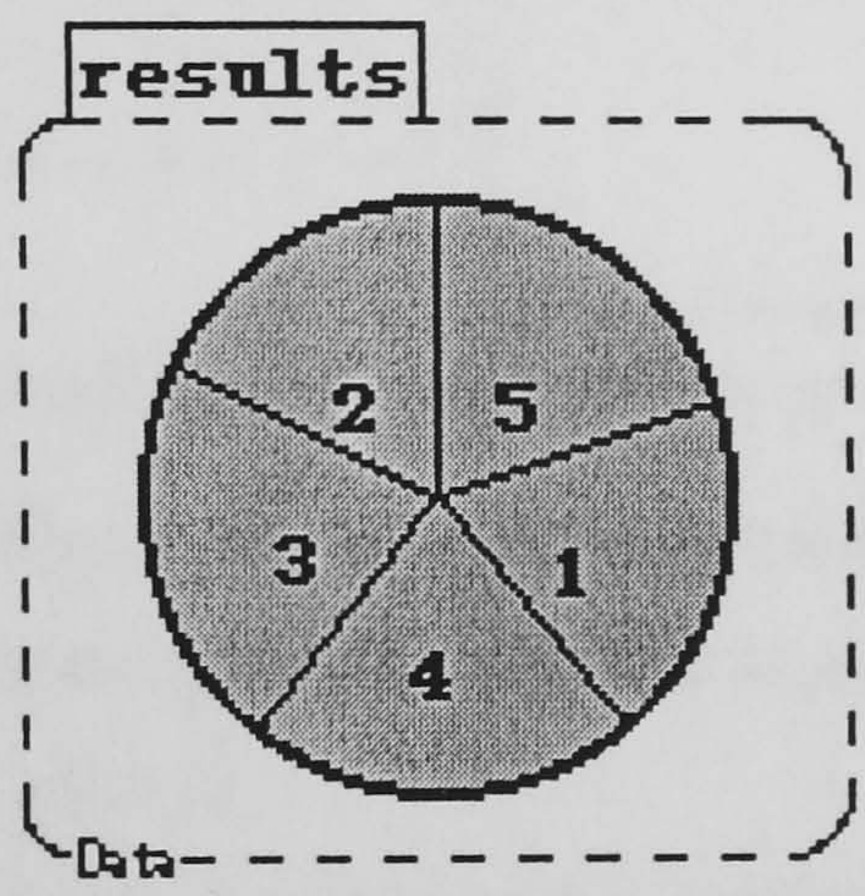

Fig. 8.13 : More even with slightly more 4's and 3's, and less 2's

I asked how they would explain this picture.

> Rebecca: "There's a higher number, so the more chance of it being even, I think .... The more times you throw it, the evener it seems to get. And I think that's because there's more chances for a number to come up than if you do it say 50 times."
>
> (6.8.7)

They then carried out a new batch of 200 trials. The pictogram indicated most 2's and least 4's. Anne suggested that they try 1000 trials (6.8.9). The pie chart showed even sectors (Figure 8.14).

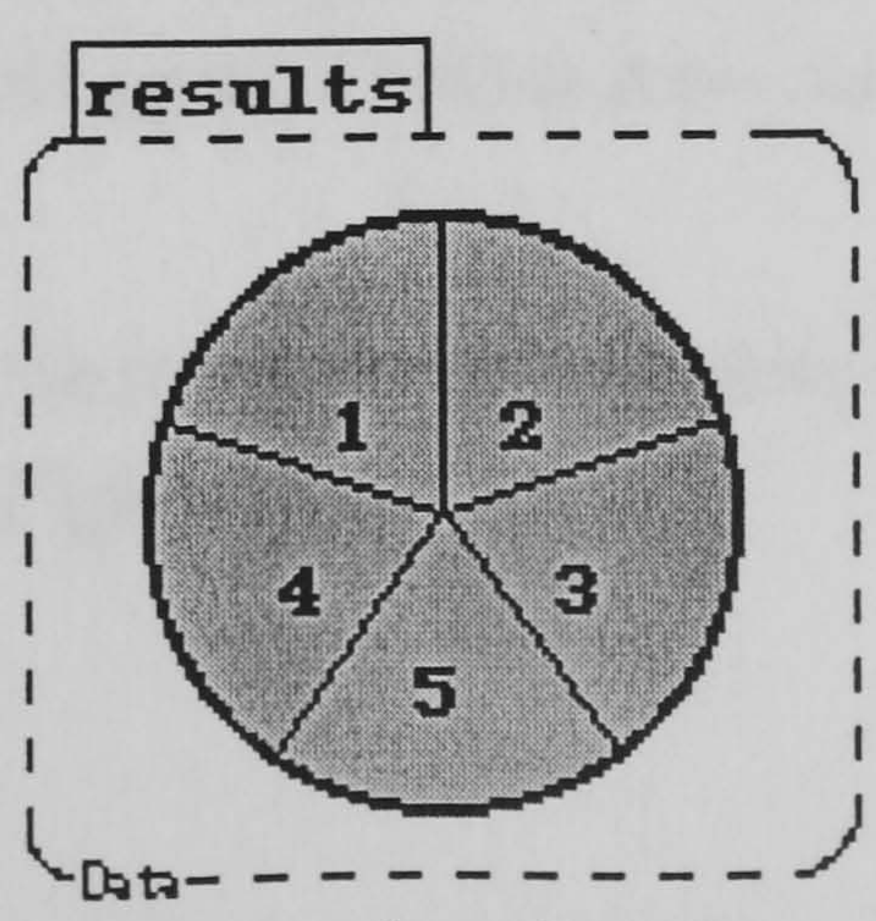

Fig. 8.14 : The pie chart showed even sectors

> Anne: "Yes, that looks even." Rebecca agrees and adds, "It looks just like the spinner does." I ask if they think the spinner is fair. They believe it is.
>
> (6.8.9)

Anne and Rebecca use the dice gadget

Anne and Rebecca then began to use the dice gadget (6.9.1) (Figure 8.15).

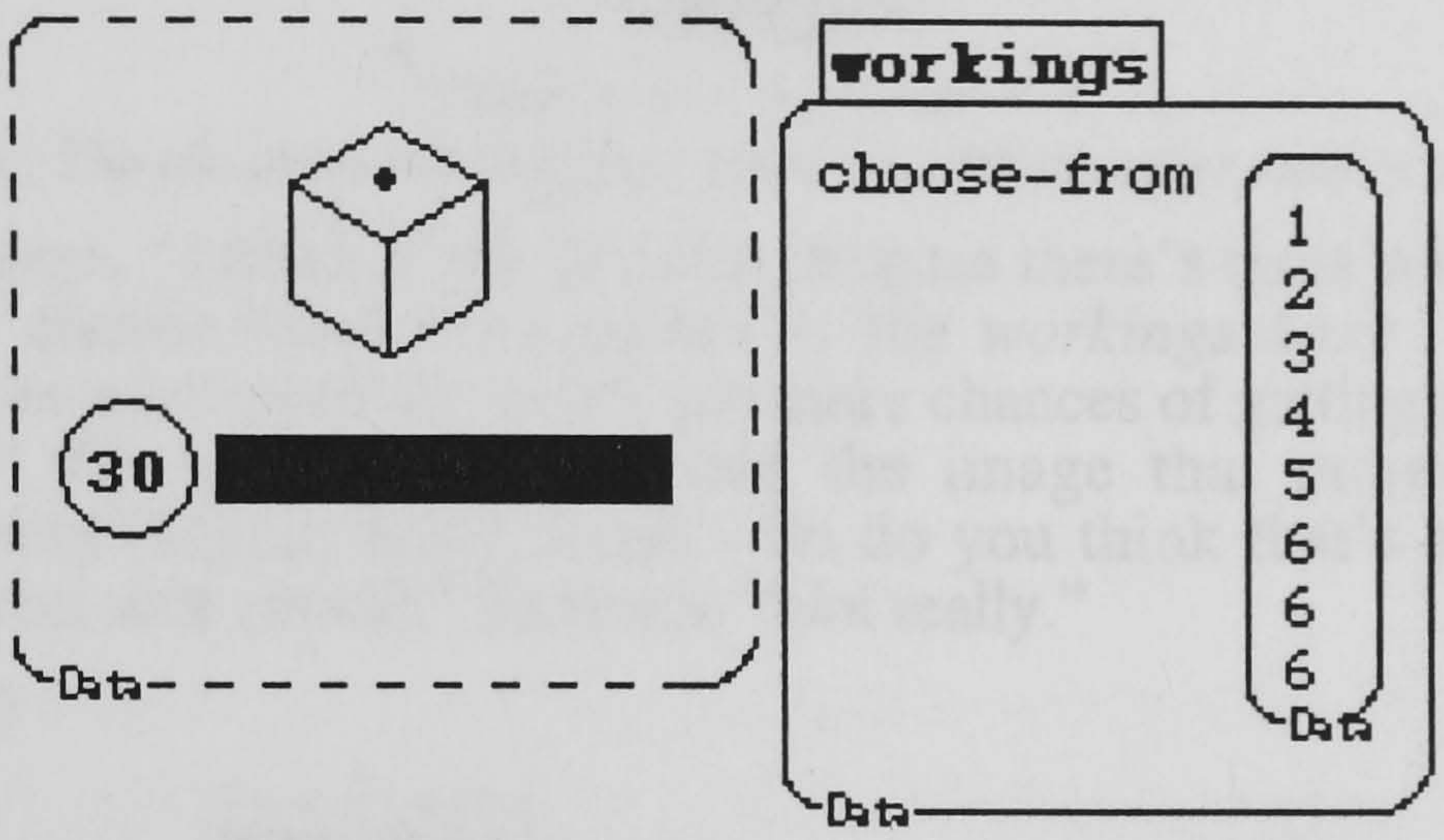

Fig. 8.15 : Anne and Rebecca then began to use the dice gadget

During the familiarisation process, they commented on how there appeared to be more 6's, and this was confirmed by looking at the results after 50 trials. They decided to repeat 1000 trials (6.9.2).

> I ask, "What's the advantage of doing it 1000 times?" Anne: "You get more and you can sort of estimate." I ask what will happen. Anne: "Maybe a bit even." Rebecca: "Maybe it's going to be even again because it seems to go more even the more times you throw it." Anne: "I think it's the more you throw it, the more even it gets." Rebecca: "Yes, that seems to be the case." Anne: "Because that's what happened most of the times, the more you get, the more even you get."
>
> (6.9.2)

Anne and Rebecca had not discriminated the conditions under which their situated abstraction applied. In a sense, they were over-generalising, not yet appreciating that the evenness of the pie chart was connected with the evenness of the workings box. In this situation, the workings box, which Anne and Rebecca had so far ignored, contained more 6's.

After 1000 trials of the dice, the pie chart showed many more 6's with the other sectors about equal (6.9.3) (Figure 8.16).

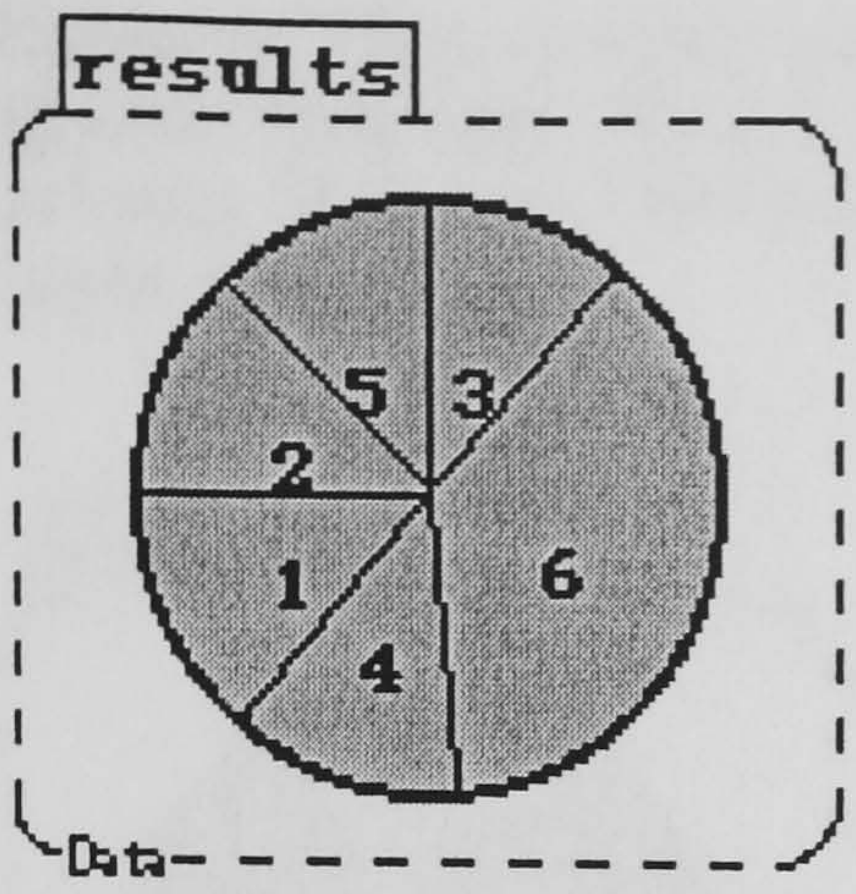

Fig. 8.16 : The pie chart showed many more 6's with the other sectors about equal

Anne says, "I think 6's is popular because there's quite a lot of 6's in the choose-from *(and points to the workings box)* .... There might be a lot more 6's so it's got more chances of getting more 6's on it." The pictogram confirms the image that more 6's are appearing (Figure 8.17). I ask, "So do you think that's behaving like a real dice should." Rebecca: "Not really."

(6.9.3)

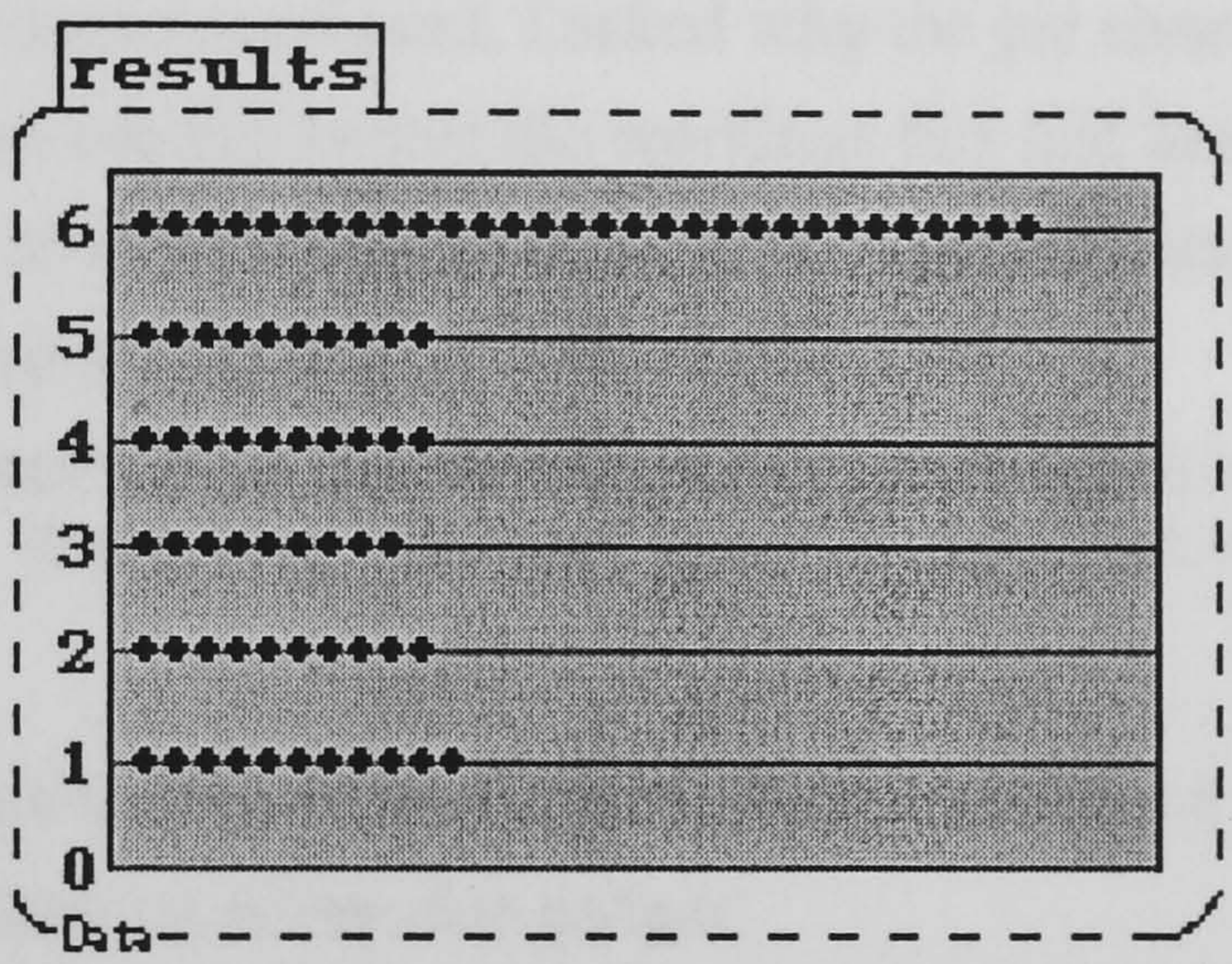

Fig. 8.17 : The pictogram confirms the image that more 6's are appearing

Anne connected the appearance of the pie chart with the data in the workings box. After some discussion about whether to use
choose-from [1 1 2 2 3 3 4 4 5 5 6 6] or
choose-from [1 2 3 4 5 6], they opted for the latter (6.9.4). They decided to generate 1000 trials (6.9.5). I asked them what it would look like.

Anne: "Very even." Rebecca: "Yes, roughly even .... I think there's more chance of getting other numbers. Well, a 50 / 50 chance." The pie chart shows even sectors. Rebecca: "Oh look, it's lovely (Figure 8.18)." Anne: "I was right again."

(6.9.5)

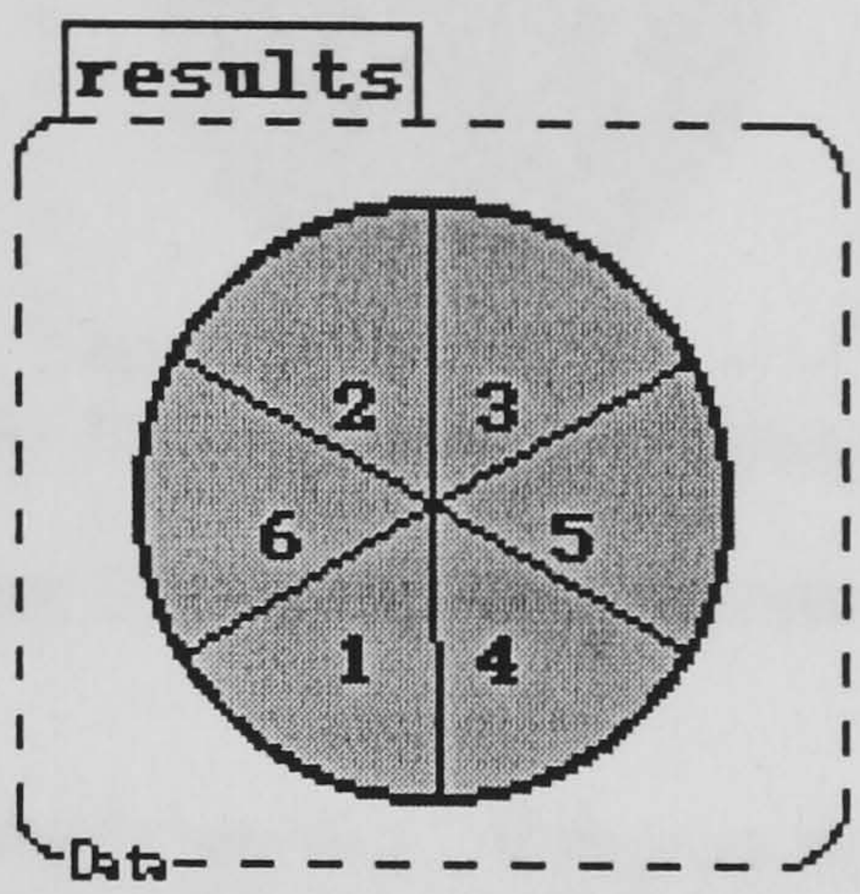

Fig. 8.18 : Rebecca: "Oh look, it's lovely""

I was in fact still interested in whether they appreciated why the pie chart had not been even in the previous experiment. I asked why the pie chart now appeared to be even when it had been uneven before the workings box had been altered. In both cases large numbers of trials had been used. After some confused attempts to explain by Anne, Rebecca offered a clearer explanation.

"Because there's more chance of getting a 6. When it stops it might land on a 6." Anne says, "Because the workings were unfair."

(6.9.5)

At this point, I decided to change the context slightly to test out their global meanings for the behaviour of the dice gadget.

"Let's say we were playing a game, and for some peculiar reason in this game, it would have to be a computer game because we are using the computer dice, we wanted there to be a good chance of getting 1's, and a fairly good chance of getting 2's but a pretty low chance of getting anything else. It's a strange game. How would we make this dice behave like that."

(6.9.6)

Rebecca edited the workings to read **choose-from [1 1 2 2 3 4 5 6]**. The two girls tested this out by repeating 1000 trials. I asked what the pie chart will look like.

Rebecca: "More 2's, more 1's and less of the others." I ask how the 1's and 2's will compare. They say they will be roughly even.

(6.9.6)

The pie chart confirmed their prediction (Figure 8.19).

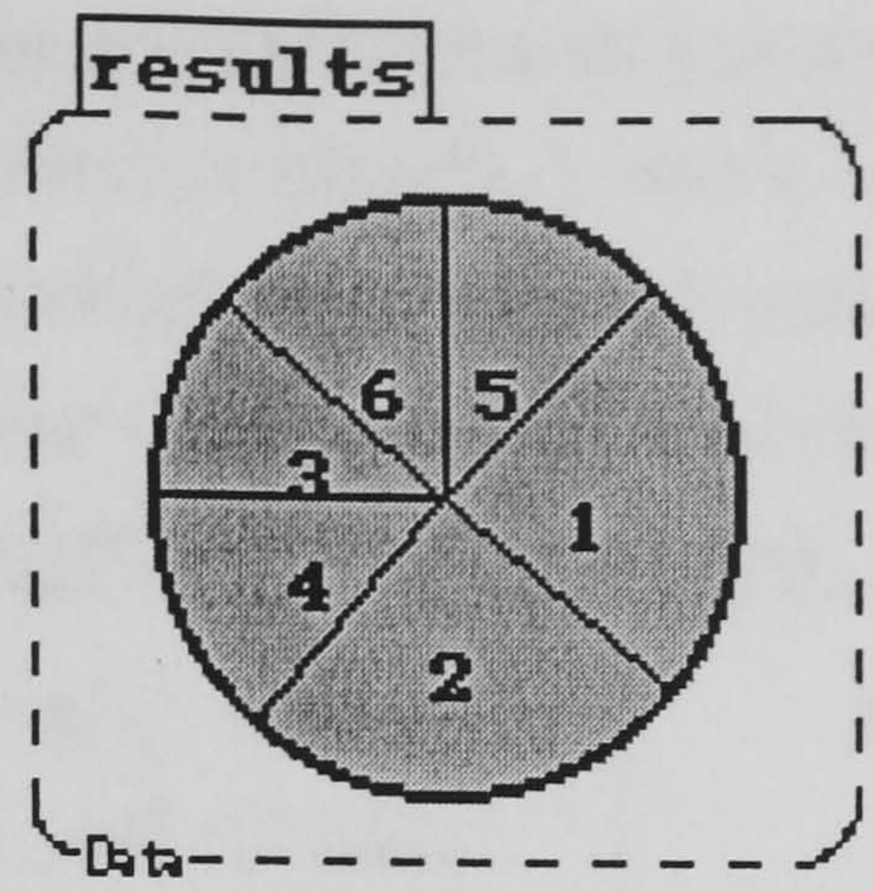

Fig. 8.19 : The pie chart confirmed their prediction

I asked (6.9.7) whether doing 50 trials would generate a similar picture. They both said not.

> Anne adds, "A bit more uneven." Rebecca says, "There'd be more 1's and 2's." Anne says, "They'd probably be about even." Rebecca says, "Maybe."
>
> (6.9.7)

I pointed out (6.9.8) that what I had really wanted was for the 1's to have a very good chance and the 2's to have only a fairly good chance. Rebecca immediately edited the workings box to read **choose-from [1 1 1 2 2 3 4 5 6]**.

Anne and Rebecca now discriminated between cases where the workings were fair, which produced even pie charts when the number of trials was high, and those where the workings were unfair, which generated uneven pie charts. They discriminated between large numbers and large small numbers, the latter generating less evenness when the workings were fair.

8.2.3. A Trace of Anne's and Rebecca's Use of the Coin, Spinner and Dice Gadgets

We can track the process by which the two girls constructed new global meanings for the long term behaviour of the coin, spinner and dice gadgets. In Table 8.1, I refer to conjectures, initial meanings and situated abstractions. Conjectures are characterised by leading and directing the path of subsequent work. They are not necessarily beliefs; quite often they represent explanations which have to be eliminated as part of the sense-making process, in which case conjectures signal a complementary belief. Initial meanings are those which the pre-interviews, or the early interactions with the Chance-Maker microworld, would suggest were already held, even before the sessions with the microworld took place (such as the local meanings). Situated abstractions follow activity, representing a meaning which has been drawn out of the activity through the forging of connections between that activity and internal resources, such as meanings already held from prior experience. Whereas conjectures are usually transient, situated abstractions tend to be maintained for longer periods of time and are formulated in terms of the activity that has previously ensued. The table below also indicates those interventions, which I see as critical in the direction of future activity.

Para	Conjectures (C), Initial Meanings (IM) and Situated Abstractions (SA)	Critical Interventions
6.7.8 to 6.7.9	The coin gadget generates more tails (C)	Is there advantage in doing more tosses?
6.7.9	*Activity : A & R carry out 100 trials and the pictogram shows more tails.*	
6.7.9	A: Tails are more popular (SA)	Would you get the same picture again?
6.7.9	*Activity : A & R do a new experiment of 100 trials. This time heads appears more often.*	
6.7.9 to 6.7.10	A: You can't really estimate (SA) R: You can't be sure (SA) R: With a real coin, it depends on how you flick it (IM)	What if we do it 200 times?
6.7.10	*Activity: A & R repeat 200 new trials. The pictogram shows roughly equal rows.*	
6.7.10	R: It may be just chance that it came out even (C) A: I think there was a reason (C)	What if we did it 500 times?
6.7.11	*Activity: A & R repeat another 1000 trials (intending to do 500). The pie chart is even.*	
6.7.12	A & R: The higher the number of trials, the more even the pie chart gets (for the coin gadget and real coins) (SA) R: With less trials, the pie chart is only half even (SA)	
6.8.1 to 6.8.2	*Activity: A & R familiarise themselves with the spinner and find that there are a lot of 1's.*	

6.8.3	The spinner generates too many 1's (SA)	What would you do to make it fair?
6.8.3 to 6.8.5	*Activity: A & R edit the workings to read:* ***choose-from [1 2 3 4 5]****. The pie chart is uneven. They edit the workings to include two of each number. After 50 trials, the pie chart is still uneven.*	
6.8.5	R: Perhaps we should take one more of the 1's away (C)	
6.8.5 to 6.8.6	*Activity: They edit the workings by deleting one of the 1's and do 50 new trials. The pie chart is still uneven.. Anne edits the workings back to* ***choose-from [1 2 3 4 5]****. and they repeat 50 new trials.. The pie chart is fairly even but not entirely convincing.*	
6.8.6	R: Maybe we should throw it more times like the coin (C)	
6.8.7	*Activity: They repeat 150 new trials.. The pie chart is more even.*	
6.8.7	R: The higher the number of trials, the more even the pie chart gets, I think(C)	
6.8.7 to 6.8.9	*Activity: They repeat 200 trials, and the pie chart is a little uneven. They repeat 1000 new trials and the pie chart shows even sectors.*	
6.8.9	R: The more times you spin it, the more even the pie chart gets (perhaps with an attached condition that the workings must be fair) (SA)	
6.9.1	*Activity: A & R begin to use the dice gadget.. There appear to be too many 6's.*	
6.9.2	A & R: If we do it a lot of times, the pie chart will be even (C)	
6.9.2	*Activity: A & R repeat 100 trials and find that the 6's are much more common.*	
6.9.3 to 6.9.4	A: There are more 6's in the results because there are more 6's in the workings box (SA)	Can you mend the dice gadget?

6.9.4 to 6.9.8	*Activity: R edits the workings to contain just one 6. They repeat 1000 trials. The pie chart shows even sectors .... I ask them to make a dice which favours 1's and 2's. They edit the workings to read:* ***choose-from [1 1 2 2 3 4 5 6]*** *and repeat 1000 new trials. The pie chart shows most 1's and 2's and least 3, 4, 5, and 6's. I ask them to make the 1's more likely than the 2's. They edit the workings to read:* ***choose-from [1 1 1 2 2 3 4 5 6].***	
6.9.6 to 6.9.8	A & R : Higher numbers of trials generate even pie charts when the workings are fair (SA) A & R : Smaller numbers of trials result in the pie chart being less even.(SA)	

Table 8.1 : A trace of Anne's and Rebecca's construction of global meanings

8.2.4. Discussion of Anne's and Rebecca's Use of the Coin, Spinner and Dice Gadgets

The trace in Table 8.1 shows us how Rebecca's and Anne's local meanings were shaped and re-shaped through the webbing with the tools and resources of the Chance-Maker microworld. This shaping process can be seen as one of co-ordination between local meanings for stochastic and deterministic behaviour, which enabled the construction of new global meanings.

There appear to be four phases through this co-ordination process. These phases do seem to have a certain progression about them, but this progression is not one in which new and better ideas replace previously established ones. On the contrary, new pieces of knowledge, situated abstractions, are constructed in such a way that their meanings are connected with, or distributed across, previous local meanings.

Sense-making through well-established local meanings

Anne and Rebecca's made sense of their early interactions with the Chance-Maker microworld through the use of local meanings. We see references to unpredictability and unsteerability.

> Anne: I don't think you can really estimate which one.
>
> Rebecca: "You can't be too sure really."
>
> Rebecca: "If you start on tails, it might land on tails again because it might not be a very good flick.
>
> Anne: "The first time there were loads of tails, so I thought it was going to be tails again. But probably after a couple of goes, it will probably do tons of heads again."

At this point in time, Anne's and Rebecca's only resources for making sense of long term behaviour was through these sorts of meanings.

The number of trials controls the evenness of the pie chart

After an experiment in which they try out 200 trials of the coin gadget, Anne conjectured that the evenness of the pie chart might not be entirely coincidence. When they observed the evenness of the pie chart for 1000 trials, they abstracted the notion that the higher the number of trials, the more even was the pie chart.

> Anne: "I think it's the highest the number, the even more it gets."

Rebecca even advanced a corollary to this situated abstraction which suggested that lower numbers of trials would be not so even.

> Rebecca: "Because the other time, when we did less numbers, it was half um even really."

It is clear in the detail of how Anne and Rebecca reached these conclusions that they have been constructed through interactions with structures which enabled the repetition of many trials (the **repeat** primitive) and the image of aggregated results (the graphing tool, especially the pie chart).

This knowledge, having been constituted through webbing with these tools and resources, was likely to be deeply connected with them, and so one might regard the abstraction as situated in the domain of the coin gadget. Just how situated, and therefore constrained, was this knowledge became apparent in the subsequent interactions with the spinner gadget.

The workings box controls the evenness of the pie chart

In many ways, the coin gadget had not been problematic, since its workings were set to cause the gadget to behave exactly like an everyday coin. The girls had therefore not needed to engage with the workings box. Indeed, there had been no reference to the workings box until they began to interact with the spinner.

Initial experiments with the spinner gadget had suggested to Anne and Rebecca that perhaps there were too many 1's appearing in the results. Consequently, a need arose for them to consider whether the workings box was causing a degree of unfairness. Anne and Rebecca were quickly able to edit the workings box so that each possible outcome appeared exactly once.

There now followed an extended period in which Anne and Rebecca carried out a sequence of experiments involving about 50 trials each time. Each experiment generated a pie chart in which the possible outcomes were quite unequally represented. From our perspective, this is not surprising since we would not expect evenness in the pie chart for only 50 trials, even if the workings were fair.

Anne and Rebecca however reacted by editing the workings to try to make the pie chart come out even. First they changed the workings to include two of each possible outcome, then, seeing too many 1's in the following pie chart. they removed one of the 1's from the workings box. When this failed to work they tried a workings box with one of each outcome represented — back in fact to where they started.

It is clear from their actions that they expected there to be a direct relationship between the entries in the workings box and the appearance of the pie chart, even though they were only using 50 trials each time.

The extraordinary thing about this episode is that they appear to have 'forgotten' the situated abstraction from the coin that 'the higher the number of trials, the more even is the pie chart'.

How might we explain Anne's and Rebecca's actions? My interpretation is that the introduction of the workings box brought this very much to the forefront of their attention. A meaning that changing the workings box would change the pie chart was easily cued by its connection to well established intuitions of deterministic behaviour. The reliability of such intuitions was far in excess of recently acquired and therefore very tentative situated abstractions, which, in any case, were connected with the various surface features of the coin gadget, not apparent for the spinner.

The number of trials AND the workings box controls the evenness of the pie chart

It would be easy to dismiss Anne and Rebecca's tinkering with the workings box as misconceived. On the contrary, we see that it is exactly these prior experiences which allow the co-ordination of a powerful situated abstraction, which deals perfectly adequately, even from our perspective, with cases where the number of trials is low or high, and the workings box is uniform or not. Note how Rebecca, having edited the workings to read **choose-from [1 1 2 2 3 4 5 6]** (6.9.6) predicts "More 2's, more 1's and less of the others" for 1000 trials, but that, for 50 trials, Anne explained, "A bit more uneven." (6.9.7)

8.2.5. A Theoretical Sketch of Anne's and Rebecca's Construction of Global Meanings

I am now able to portray the co-ordination of meanings by which Anne and Rebecca forged new connections about the long term behaviour of the coin, spinner and dice gadgets through a tentative sketch (Figure 8.20), which will be later modified to incorporate evidence from the other case studies.

The sketch schematises the process by which new connections, in the form of situated abstractions, are forged out of juxtaposition of weakly connected local meanings with external structures within the Chance-Maker microworld. Thus meanings such as unsteerability are reserved for cases where the number of trials is low and new causal meanings for control emerge to make sense of long term behaviour. Further co-ordination of causal influence of the number of trials and the workings box is also depicted in the emergence of a meaning in which they are combined.

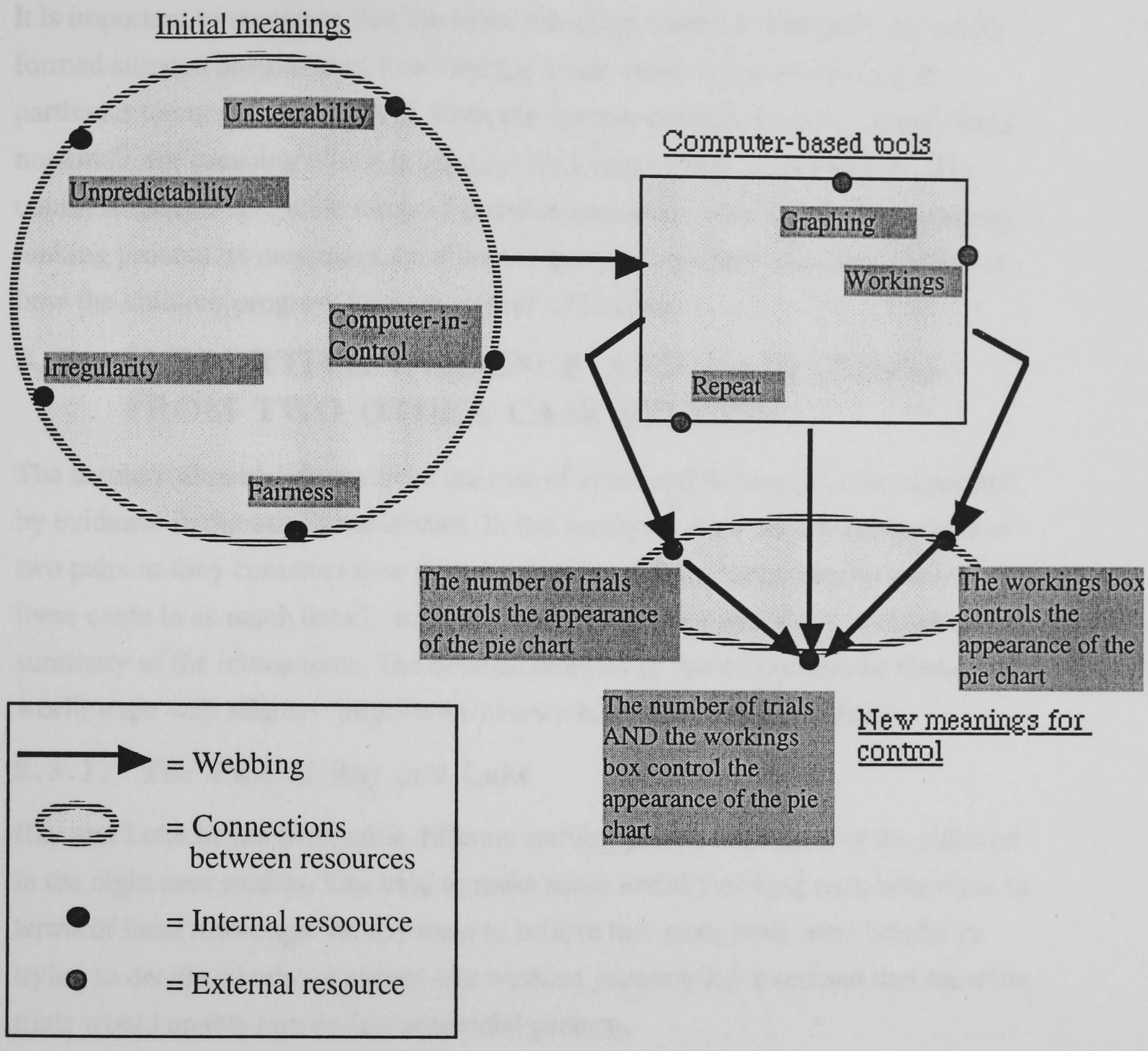

Fig. 8.20 : A sketch of Anne's and Rebecca's construction of global meanings

I want to help the reader to make sense of Figure 8.20. Let's start in the top left corner. The five local meanings are depicted as contained within a closed set to illustrate the connections, such as interchangeability, between those meanings. As the children interact with the Chance-Maker's gadgets, one or more of these local meanings are cued by features of that activity, as depicted by the webbing arrow connecting the set of local meanings to the box of computer-based tools. Out of this connection, emerge new meanings for control, in the form of the two situated abstractions: 'the number of trials controls the appearance of the pie chart' and 'the workings box controls the appearance of the pie chart'. Further activity with the computer-based tools allows the co-ordination of these two situated abstractions into a third: a global meaning, which states 'the number of trials AND the workings box control the appearance of the pie chart'.

It is important to recognise that the initial meanings continue alongside the newly formed situated abstractions, available for future sense-making activities. In particular circumstances, such as when the number of trials is neither clearly large nor small, the meaning which is actually cued, may appear quite arbitrary. The continued access to a wide range of possible meanings, which gives the meaning-making process its messiness, confounds attempts to present a smooth picture of how the children progress from one 'level' to another.

8.3. SUPPORTIVE EVIDENCE AND VARIATIONS FROM TWO OTHER CASE STUDIES

The theoretical model drawn from the case of Anne and Rebecca can be supported by evidence in the other case studies. In this section, I draw on the interactions of two pairs as they construct new global meanings. It is not necessary to present these cases in as much detail, using the trace of how new meanings evolved as a summary of the interactions. The detailed analysis of these cases can be found at the world wide web address: http://www.warwick.ac.uk/wie/staff/DP.htm.

8.3.1. The Case of Ray and Luke

Ray and Luke began from quite different starting points. Like most of the children in the eight case studies, Ray tried to make sense initially of long term behaviour in terms of local meanings. He did seem to believe that more trials were helpful in trying to decide whether a gadget was working properly but it seemed that the extra trials would enable him to find sequential patterns.

> "The more you do it, you may be able to understand the pattern in it."
>
> (2.6.2)

Again later, when I asked whether 500 trials were advantageous over 5 trials, Ray referred to sequential patterns.

> "Not particularly ...maybe, maybe, yes. It may not be able to get the full pattern."
>
> (2.7.2)

Luke on the other hand did seem to have some global meanings. For example, the boys generated 100 trials of the coin gadget and inspected the pie chart, which showed roughly even sectors for heads and tails (Figure 8.21).

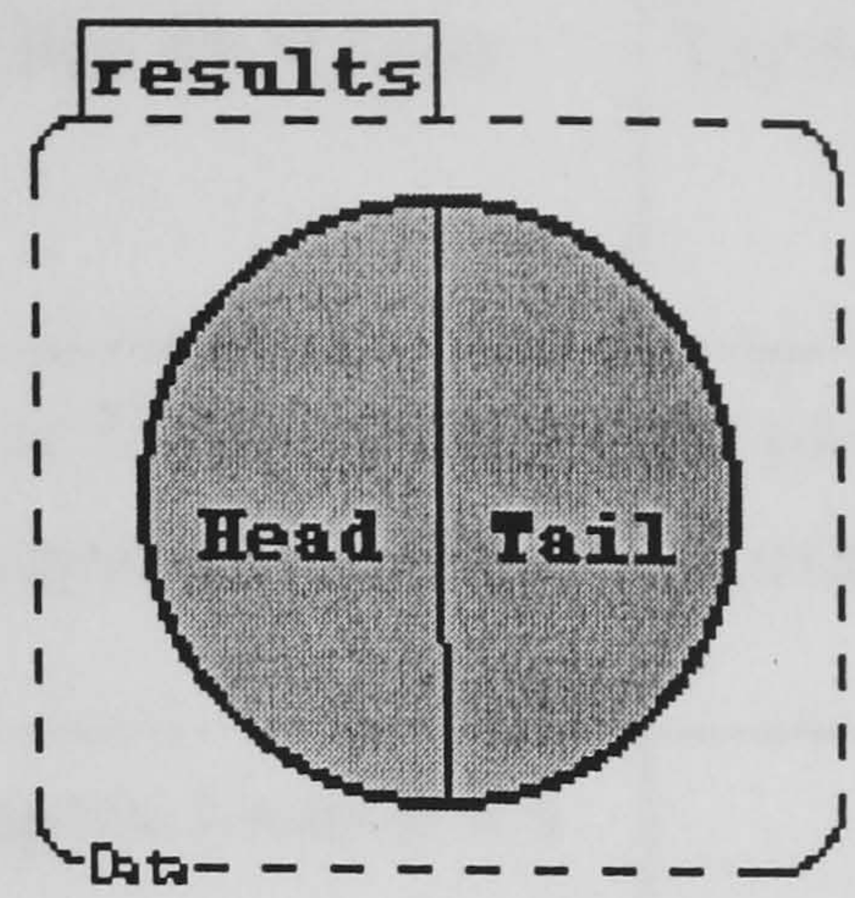

Fig. 8.21 : The pie chart showed roughly even sectors for heads and tails

Luke thought that this was reasonable. I asked what would not be a reasonable picture for a real coin.

Luke replies, "One quarter tails ...or exactly half." (2.6.3)

Ray's and Luke's subsequent interactions are summarised below.

Para	Conjectures (C), Initial Meanings (IM) and Situated Abstractions (SA)	Critical Interventions
2.7.2	R: Large numbers of trials allow the sequential pattern to emerge (IM) L: Large numbers of trials allow all possibilities to be represented (IM)	
2.7.2 to 2.7.4	*Activity: R & L repeat 500 trials of the spinner. The pie chart shows about half of its area is taken up by 1's. Both boys notice that the pie chart looks like the spinner itself.*	
2.7.5		Can you make the spinner random?
2.7.5 to 2.7.6	*Activity:: R & L edit the workings to contain just one of each possible outcome. They repeat 1000 trials (intending in fact to do 500). The pie chart is pretty well even.*	
2.7.6 to 2.7.7	The spinner is random because the pie chart is even (SA)	Can you make the spinner so that it is slightly harder to get a 5?
2.7.7 to 2.7.8	*Activity: R & L edit the workings to contain two of each number but only one 5. They repeat 500 trials and predict that the pie chart will look like the spinner. This prediction proves to be correct.*	

2.7.8 to 2.7.9	The pie chart looks like the spinner (SA)	Try the experiment for just 20 trials.
2.7.9	*Activity: They repeat 20 new trials and the pie chart shows more 5's than expected. Another experiment of a new 20 trials still generates more 5's than expected.*	
2.7.9	R: Anything can happen because it's random (IM) L: Perhaps it's the strength or the position of the 5 that's causing there to be too many 5's (C)	
2.7.9	*Activity: They repeat a new 20 trials. 5's are low but less 2's and about as many 5's as 3's and 1's.*	
2.7.9 to 2.7.10	R: The number of 5's in the workings box is so close to the others that it is not surprising that this happens. (C)	What would we need to do to make the pie chart come out more even?
2.7.10	*Activity: R & L edit the workings to increase the number of 1's, 2's, 3's and 4's to three of each whilst leaving just one 5. After 20 new trials, the pie chart shows least 5's but not by very many. R believes that this confirms his conjecture but L argues that there is not much difference and that the 5's would be just as frequent if they added a 5 back into the workings. This they do. They repeat 20 new trials and the pie chart shows even less 5's than in the last experiment.*	
2.7.11	R: Because it is random, you can not predict what will happen whatever the number of trials (IM)	
2.7.11 to 2.8.2	*Activity: They repeat 500 trials and the pie chart shows the sectors to be about equal except that the 5 is slightly less. They move on to work with the dice and notice that the workings have three 6's. They expect there to be more 6's in the results. They repeat 1000 trials to test this out.*	

2.8.2 to 2.8.5	L: With 20 trials, the results can not be shared out properly between the different possibilities (C) L: With a large number of trials it might be more even (C) R: The higher the number of trials, the more chance of finding a pattern (SA)	Can you make the dice random?
2.8.5	*Activity: R & L edit the workings to include one of each number. They repeat 1000 new trials. The pie chart is about even.*	
2.8.6		Do you think the pie chart would also be even for 20 trials?
2.8.6	L: The pie would be different for 20 trials because 20 does not divide by 6 (C) R: When you do it a smaller amount of times, it doesn't always come out the same (SA)	Try 18 trials instead
2.8.6 to 2.8.7	*Activity: L says there will be about three results on each. R & L repeat 18 new trials and the pie chart shows large variations. L is surprised. They try a new 20 trials and the pie chart is different again. L supposes that it is because 20 does not divide by 6 and so they try 12 trials. The pie chart is still very uneven and different again.*	
2.8.7	It is unpredictable because it is random (IM)	Why did it not come out uneven when we did 1000 trials?
2.8.7	*Activity: Ray says, only half seriously, that the computer likes 1000 trials and so I suggest that they try 1001 trials. They repeat 1001 new trials. The pie chart looks even.*	
2.8.7 to 2.8.8	When the number of trials is small, some outcomes lag behind. When you increase the number of trials, those numbers have the chance to catch up. (SA)	Suppose I want a dice where it is quite hard to get a six, very easy to get a 1, and anything else moderately easy

2.8.8 to 2.8.9	*Activity: R & L edit the workings to include four 1's, three of each of 2, 3, 4 and 5, and just one 6. They repeat 1000 new trials and the pie chart confirms their predictions which were consistent with the workings. I ask them if the pie chart would look the same for 20 trials. They say not and argue that when the number of trials is small, some numbers can lag behind.. They repeat 20 trials and the pie chart does indeed indicate unpredictability.*

Table 8.2 : A trace of Ray's and Luke's construction of global meanings

Ray's co-ordination was not dissimilar from that of the previous cases. He began by using local meanings, in particular irregularity, and did not believe that extra trials would afford any new information regarding the correct working of the gadgets. He soon however reached a point where, at least for the spinner, he was able to predict the evenness of the pie chart for a large number of trials. Small numbers of trials were simply random and he had to wait rather patiently while Luke came to terms with the behaviour for large small numbers. Nevertheless, this experience gave Ray a meaning for the transition from small to large numbers, and this meaning drew directly from his irregularity meaning. Short term behaviour was random because some outcomes had not had the chance to appear, but, in the longer term, the pattern played itself out and all numbers had the chance to catch up.

Luke's co-ordination of meanings was quite different from the earlier cases and that of Ray. He began with a meaning, which in hindsight was closely connected to fairness. If a gadget was unfair then he had few problems in predicting the appearance of the pie chart for both large and small numbers of trials. When though the gadget was fair, he felt that the results should be equally spread between the outcomes even when the number of trials was small. Indeed, when the number of trials would not divide equally between the number of different outcomes, then he used this to explain the chaotic appearance of the pie chart. Eventually Luke was able to converge with Ray's meaning for the transition between short term and long term behaviour.

8.3.2. The Case of Donna and Rose

Both Donna and Rose relied heavily on the unpredictability meaning for randomness. Thus, when the pie chart for the coin seemed to change with every experiment (using no more than 100 trials), Rose explained, "It changes. It's not always going to be the same" (3.7.3). The girls' subsequent interactions are summarised below.

Para	Conjectures (C), Initial Meanings (IM) and Situated Abstractions (SA)	Critical Interventions
3.7.6	*Activity: After 100 trials of the coin gadget, the pie chart shows more tails than heads.*	
3.7.7	R: Doing a few more trials may make the pie chart even (C) D: Increasing the strength may make the pie chart more even (C)	
3.7.7 to 3.7.8	*Activity: D & R replicate their experiment, using 100 trials again, but with the strength set to 100. The pie chart is a little more even.. Wondering whether the increased strength brought about the improvement, they decide to try a strength of 50*	
3.7.8		Earlier, Rose, you suggested more trials may make the pie chart more even.
3.7.9 to 3.8.2	*Activity: D & R try another 110 trials and the pie chart looks even. After another 115 trials, making 325 in all, the pie chart is a little less even.. They turn to the spinner gadget and decide to try 100 trials.*	
3.8.2	150 trials may be better than 92 because you have more chance of seeing (C)	
3.8.2 to 3.8.3	*Activity: D & R accidentally do 300 trials. The pie chart shows 1's as roughly half of the pie. The result is not surprising for the girls.*	
3.8.4		Can you make the spinner fair?
3.8.5 to 3.8.6	*Activity: They edit the workings to chose from one of each number from 1 to 5.*	
3.8.6	D: The pie chart will look like the spinner, equally divided (C)	
3.8.6	*Activity: They intend to repeat 150 trials but do 300 by mistake. The pie chart for 300 trials shows almost equal sectors..*	

3.8.7 to 3.8.8	300 trials will be more useful than 150 because you have more to go on (SA)	What will happen if you do 5 trials?
3.8.8	*Activity: They predict that the pie chart will appear more even still. In fact, the pie chart for 5 trials is not at all even, with some sectors missing completely..*	
3.8.8	The smaller the number of trials of the coin gadget, the bigger the variation in the sectors of the pie chart (SA)	
3.9.1 to 3.9.3	*Activity: D & R move on to the dice gadget. They immediately edit the workings to make the dice fair. After considering doing 100 trials, they decide to do 6. They obtain an uneven pie chart with some possibilities missing.*	
3.9.3 to 3.9.4	D: If we do it more times, anything could happen; it is unpredictable (IM) R: The dice gadget may be like the spinner: more trials may make the pie chart more even. (C)	If I only let you have one go at this, how many times would you repeat the experiment?
3.9.4 to 3.9.5	*Activity: R suggests it is possible to have too many trials. They decide to repeat 60 new trials, having rejected 100 trials because there would be 40 trials remaining after 10 had been allocated to each of the six possibilities (or sectors). The pie chart shows more 5's and 3's. 5's have now come out most twice in succession.*	
3.9.5	The results are influenced by the positions on the dice. Is 5 opposite 3? (C)	I explain how a dice is constructed.
3.9.5	The strength influences the outcomes (C)	
3.9.6	*Activity: They change the strength to 30 and repeat 60 trials. The pie chart shows many fewer 5's.*	
3.9.6	The particular number of trials influences the outcomes (C)	
3.9.6	*Activity: They try 50 trials instead of 60. The pie chart shows most 3's and still looks uneven.*	
3.9.7		Why did you, Donna, suggest using 600 trials earlier?

3.9.7 to 3.9.8	*Activity: This intervention leads to them trying out 600 trials. The pie chart looks fairly even.*	
3.9.9		Try 1000 trials
3.9.9	*Activity: D & R repeat 1000 new trials. The pie chart looks very even.*	
3.9.10		If you only did it 500 trials, would it be so even?
3.9.10	*Activity: They repeat 500 trials and find that the pie is pretty even but not quite so much as had been the pie chart for 1000 trials. They continue by trying 100 trials and find it is less even still.*	
3.9.10	Like the spinner, when we do it less times, the pie chart is not as equal (SA) When you repeat more trials, the pie chart is more equal (SA)	

Table 8.3 : A trace of Donna's and Rose's construction of global meanings

At various times, Donna and Rose conjectured the following possible reasons or descriptions for the behaviour that they observed:

- Changes in strength caused variations in the appearance of the pie chart,
- The gadget was simply random and so completely unpredictable,
- The number of trials needed to be shared out equally between the possible outcomes,
- The lower the number of trials, the more variation between the sectors in the pie chart,
- The pie chart looked like the spinner,
- When they did it more times, the pie chart came out more equal.

The Chance-Maker microworld gave Donna and Rose the sorts of control mechanisms that allowed them to test out these conjectures. The strength control allowed them to try out different strengths and try to find a connection between the results and the strength. The ability to see the results aggregated in the form of a pie chart allowed them to look for patterns which might refute the suggestion that the system was completely unpredictable. Control over the number of trials and the workings box eventually enabled them to co-ordinate new meanings such as the last three in the list above.

These meanings did not replace the earlier meanings; they emerged to become available along with the earlier meanings for future sense-making. This much is clear from the way that Donna and Rose constantly referred back to older meanings whenever something unexpected happened.

Donna and Rose also help us to understand how deeply situated are those meanings. The experiences with the coin seemed to have little influence over their thinking about the spinner. Eventually, a link was made between the dice and the spinner but only after considerable experimentation with the dice. Indeed, even towards the end of this episode, Rose, in the face of contradiction from Donna, conceded, "I suppose the spinner's a different thing."

8.4. COMPARISON OF THE THREE CASE STUDIES

Three case studies have been presented which demonstrate the webbing between local meanings for stochastic and deterministic behaviour and the external resources in the Chance-Maker microworld, guided by my own interventions, and resulting in the co-ordination of new global meanings, associated with long term behaviour.

These new meanings take the form of situated abstractions, articulated in terms of actions within the Chance-Maker microworld and language which leans heavily on the embedded tools and structures.

8.4.1. Situated Abstractions in the Three Case Studies

In this section, I look briefly at some of the situated abstractions across the three case studies. One feature of these situated abstractions is how the language refers directly to structures within the microworld.

"The pie chart looks like the spinner"

Several children observed the similarity between the pie chart and the spinner when large numbers of trials with the spinner were carried out. Here are some examples (the parentheses are my own):

> Rebecca: "It (the pie chart) looks just like the spinner does."
>
> Donna: "That's (the spinner's) basically what I think it's (the pie chart's) going to look like."

For some children, this situated abstraction became a sort of heuristic in the sense that they would predict what the pie chart would look like on the basis of what the spinner looked like. This was often a point at which I re-problematised the situation by asking the children to consider, say, 20 trials or by moving onto the dice gadget, since the dice did not carry the same cue from its appearance.

It seems that the spinner provides the sorts of webbing opportunities which allowed relatively easily-extensible situated abstractions whilst the coin was less effective in supporting sense-making within the spinner. One conjecture is that the spinner offered two features not present in the coin:

- The spinner provided a direct link between the workings box and its physical appearance. When the workings box changed, the spinner's appearance changed, signalling the relationship between the make-up of the workings box and the fairness or otherwise of the gadget. This linkage was further extended by discoveries such as **choose-from [1 1 2 2 3 3 4 4 5 5]** was functionally the same as **choose-from [1 2 3 4 5]**. In other words, the proportionate control mechanism of the workings box was more visually accessible.
- The spinner was set to a default which was unfair. The lack of fairness built into the spinner attracted interaction with the workings box in a way that the 'fair' coin did not. If this conjecture is right, then the manipulation of the workings box becomes a central and fundamental activity in supporting the children's sense-making.

"The higher the number of throws, the more even is the pie chart"

This was perhaps the most commonly expressed situated abstraction. Here are some examples:

> Anne: "I think it's the more you throw it, the more even it gets."
>
> Rebecca: "The more times you throw it, the evener it seems to get."
>
> Rose: "When we did the spinner, when we did the higher number, it came out more equal."
>
> Donna "When we did it 60 times, it only had a chance to see how many it would get out of ten each, so I think it was easier if it did it out of 100."

Ray and Luke had their own idiosyncratic way of articulating the tendency for the pie chart to become more even with more trials. They appealed instead to Ray's local meaning based on the sequencing of results.

> Ray: "The higher the number, the more chance of finding a pattern."
>
> Ray: "Because maybe when you do a small amount, the first seven it could all come out on 6, but if you do more it could catch up with all the numbers, because there are more numbers."
>
> Luke: "On 20, you might get, because there's only a couple of throws, you might get more on a six, but, because there's a lot of throws, it might be more even."
>
> Luke: "Because on a big number, like I said, it could be easy to get sixes sometimes, sixes could come out, then it could be hard to catch up, because like there could be only two left."

The situatedness of these statements is apparent in the detailed analyses by the way that the children often needed to re-construct the situated abstraction when interacting with a different gadget. Nevertheless, connections were eventually made indicating that a situated abstraction constructed from webbing with one gadget is not completely inaccessible as a sense-making resource when interacting with a new gadget. However, it is clear that it was normal for these children to consider other possible explanations for the new gadget's behaviour, including aspects of its surface features, before connections with the previous gadgets were made. The amount of sense-making activity with the new gadget before the connection was made varied quite considerably from pair to pair.

"The smaller the number of throws, the less even is the pie chart"

The children often reached a situated abstraction, which was the converse of "the higher the number of throws, the more even is the pie chart". Instead (and often as well as), they referred to a situated abstraction in which the effect of reducing the number of trials was articulated, "the smaller the number of throws, the less even is the pie chart". Here are some examples:

> Rose: "The smaller the number you do it, the bigger some of the other numbers."
>
> Rose: "When you did it smaller, I don't think it is as equal."

Again Ray and Luke had their own way of expressing this idea:

> Ray: "A smaller number may make it not get all the pattern."
>
> Ray: "Because with a smaller number, the pattern does not come out."

Some children reached this notion first by starting with a large number of trials and then noticing the loss of uniformity as they used less trials. There were also cases where this situated abstraction was expressed alongside the previous one, indicating a co-ordination between the two.

> Rose: "When we did the spinner, when we did it less times, it didn't come out as equal, but when we did it more times, it came out more equal."

These situated abstractions did not emerge automatically simply because the children worked with the Chance-Maker microworld. On the contrary, they made many conjectures as to what might be causing the strange behaviour, especially when the number of trials was not large. The move to trying out a larger number of trials was often the consequence of running out of other possible explanations and a timely intervention from myself. The next section summarises these various conjectures.

8.4.2. Conjectured Explanations for Large Small Number Behaviour

After some top-level play with the gadgets, it was common for the children to repeat 20 or so trials as part of their strategy for deciding whether the gadget was working properly.

Four factors motivated moves to using more trials:

- *Accident*

  It was often a happy accident that the children executed the **repeat** line twice, so that instead of carrying out say 30 trials, they would in fact do 60.

- *Curiosity*

  It was sometimes the case that the children simply used the repeat structure because it was there and they were curious to see what effect it would have.

- *More information*

  It was regularly the case that the children would express a vague feeling that more trials would provide more information and so put them in a better position to decide whether the gadget was working properly.

- *Elimination of zero occurrence outcomes*

  A common feature was that a small number of trials would not be sufficient for all the possible outcomes to occur. More trials were then needed to ensure that, in practice, all possible outcomes had occurred and, in that sense, the gadget was working properly.

- *Intervention*

  When none of the other stimuli were encouraging the children to use more trials, or when they seemed to be going over ground already covered several times (such as testing again whether the strength was causing the unpredictable outcomes), I would, as a last resort, intervene and ask directly whether more trials might be helpful.

The effect of one or other of these stimuli would usually be that the children would begin to try a few more trials. More trials usually meant to them 5 or 10 more, though gradually there was a tendency to build up to much larger numbers of trials.

Typically the children would reach a point where the number of trials was a large small number, a number where there would still in practice be rather a large amount of variation in the aggregated outcomes of the gadget, but large enough for the children to believe that the intrinsic fairness of the gadget should show itself in the results. Occasionally children would leap to a high number of trials, often out of curiosity, and I would need to intervene later to problematise the situation by suggesting that they tried, say, 20 trials to see if the same patterns appeared.

The children often found difficulty in constructing meanings for the behaviour of the gadgets for large small numbers of trials. All the outcomes had the chance to occur for these numbers of trials but, somehow, they were not resulting in fair-looking pie charts, even when the workings were transparently fair.

In this situation, the children conjectured many possible reasons for the gadgets' behaviour:

- Amending the entries in the workings box would cause corresponding changes in the appearance of the pie chart,
- Changing the size of the strength might affect the pie chart's appearance,
- The position of the numbers on the gadget affected the appearance of the pie chart,
- The number of trials would not share out evenly between the possible outcomes.

The structures in the Chance-Maker microworld allowed each of these conjectures to be examined to the satisfaction of the children. So, the impact of the workings box was tested by sometimes prolonged cyclical processes, in which the children first tinkered with the workings box and then observed the resulting pie chart. The effect of the strength was tested, often quite scientifically, by varying its size whilst keeping the number of trials and all else the same. The order of the numbers in the workings box was altered to test out whether their position affected the outcomes. The irrelevance of the precise size of the number of trials could be seen in how 1001 trials produced the same overall image as 1000 trials. In contrast, 18 trials for the dice was seen to be no more effective at generating an even pie chart than had been 20 trials.

8.5. A MODIFIED THEORETICAL SKETCH OF THE CONSTRUCTION OF GLOBAL MEANINGS

New meanings in the form of situated abstractions emerged out of the refutation of various deterministic explanations. Whereas local meanings remain at the level of that which can not be explained deterministically, the ultimate irony is that the central meanings constructed for long term behaviour, despite being later developments, turn out after all to have features associated with deterministic behaviour: for example, the pie chart is even *because* the number of trials is large (provided that the workings are fair). A situated global meaning is constructed through the forging of connections between local meanings, such as fairness, unsteerability and irregularity together with the abstractions of prior experiences with cause and effect linkages, and the external structures, in particular the **repeat** primitive, editing of the workings box, and the graphing tool, especially the pie chart.

In Figure 8.2, the webbing process is sketched out in a modified version of the earlier sketch based on Anne's and Rebecca's case. The focus on control issues has been broadened to incorporate the co-ordination of fairness, unpredictability and irregularity.

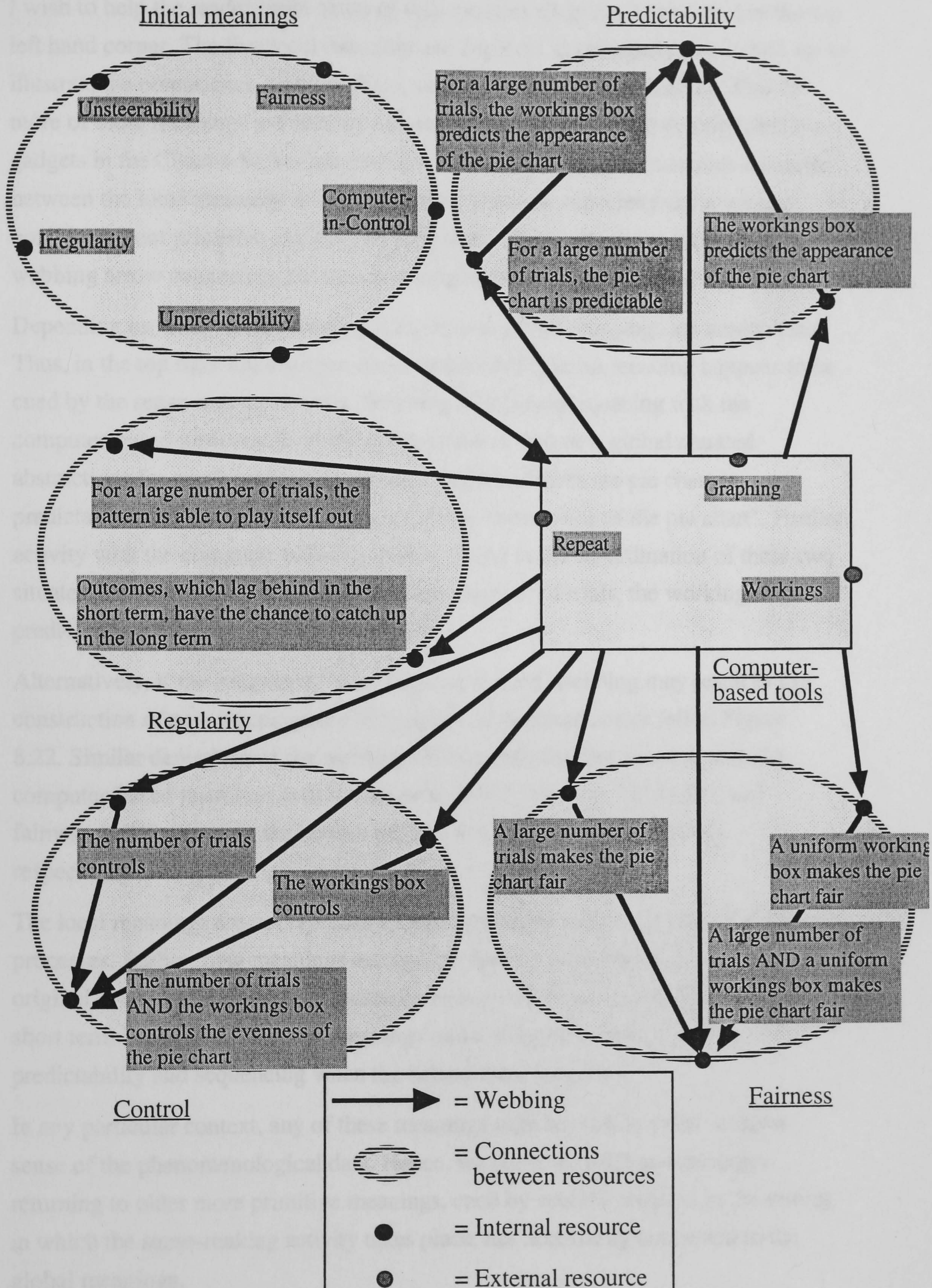

Fig. 8.22 : The construction of global meanings

I wish to help the reader make sense of this complex diagram. Let us start in the top left hand corner. The five local meanings are depicted as belonging to a closed set to illustrate the connections between them, such as their interchangeability. One or more of these meanings are cued by features of the sense-making activity with the gadgets in the Chance-Maker microworld. As a consequence connections are made between the local meanings and the external resources, especially the workings box, the **repeat** primitive and the graphing tools. This connection is depicted by the webbing arrow connecting the local meanings to the computer-based tools.

Depending on which local meaning is cued, new global meanings are constructed. Thus, in the top right hand corner, the unpredictability local meaning happens to be cued by the sense-making activity. Webbing of this local meaning with the computer-based tools results in the construction of two new global situated abstractions for predictability: ‘for a large number of trials the pie chart is predictable’ and ‘the workings box predicts the appearance of the pie chart’. Further activity with the computer-based resources results in the co-ordination of these two situated abstractions into a third: ‘for a large number of trials, the workings box predicts the appearance of the pie chart’.

Alternatively, if the irregularity local meaning is cued, webbing may result in the construction of new meanings for irregularity, as depicted centre left in Figure 8.22. Similar depictions of the webbing of unsteerability and fairness with the computer-based resources to construct new global meanings for control and fairness are illustrated in the bottom left and bottom right of the diagram respectively.

The local meanings are not *replaced* by more powerful understandings of stochastic processes. Rather, new meanings emerge, perhaps as modified copies of the original meanings. The original meanings remain to be used especially in cases of short term behaviour. The new meanings make sense of control, fairness, predictability and sequencing when the behaviour is long term.

In any particular context, any of these meanings may be cued in order to make sense of the phenomenological data. Hence, we observe children sometimes returning to older more primitive meanings, cued by specific features in the setting in which the sense-making activity takes place, not necessarily connected to the global meanings.

8.6. SUMMARY OF CHAPTER EIGHT

Local meanings focus on the immediate, trying to make sense of the similarities or variations of outcomes of the current trial and that following, anticipating the outcomes of the next few trials or reflecting backwards on the most recent trials. Local meanings are concerned with explanations or descriptions for what occurs locally (in time) and therefore make reference to factors which seem to bear upon that short term behaviour. In contrast, global meanings focus on properties which reveal themselves through aggregated results over many trials. Global meanings try to make sense of the trends and invariance in the outcomes over large numbers of trials. They are concerned with explanations or descriptions for what occurs in aggregation and therefore make reference to factors which seem to impinge upon that long term behaviour.

This distinction however does not give sufficient recognition to the part played by the local meanings in constructing global meanings. This chapter has demonstrated that global meanings emerge out of the webbing of local meanings and the specific tools and resources available in the Chance-Maker microworld, influenced by prior knowledge of cause and effect linkages.

Whereas local meanings tend to be self-explanatory and arise out of the failure of deterministic explanations, global meanings arise out of a shaping of local meanings, whereby the limitations on unpredictability, unsteerability and irregularity become transparent through the interactions with the Chance-Maker microworld. A functional view emerges in which long term behaviour is seen as increasingly fair, increasingly predictable and increasingly controllable as the number of trials increases. It is as if the long term behaviour is almost deterministic, except that short term and long term behaviour are not discrete entities, since one merges into the other. In the region of large small numbers, problems remain since it is unclear whether such situations should cue local or global meanings.

Another problematical area is how to deal with compound events, such as the total of two dice. Fairness is often seen as an attribute of the make-up of the random device itself; we can see and feel the fairness of a dice in its symmetry but we can not see or feel the total of two dice. Fairness, when cued by the appearance of the dice, may suggest that the totals for two dice will also be equally frequent. The next chapter studies children's construction of global meanings for the behaviour of compound events.

CHAPTER NINE

Meanings for Compound Events *In* a Domain of Stochastic Abstraction

9.1. INTRODUCTION

This chapter is the third in a sequence of three which examine meanings for stochastic behaviour during tool use in Iteration 3. In this chapter, the focus changes to consider the children's meanings in the domain of abstraction for *compound* events. By compound events, I refer to outcomes, which are built out of the combination of simpler events. In the Chance-Maker microworld, such compound events are encountered when interacting with the two-spinners and two-dice gadgets, which both focus attention on the totals of two random generators. For example, a total of 4 is a compound event since it can be deconstructed into the simple events: 1+3, 2+2 and 3+1.

As outlined in the overview of Chapter Seven, the broad approach is to describe, categorise and explain different meanings. We have seen in Chapter Eight that the construction of global meanings emerges out of, and is co-ordinated with, local meanings for stochastic and deterministic behaviour (themselves abstracted from everyday experience) through interaction with the tools and structures within the Chance-Maker microworld. In particular, I reported on evidence of the co-ordination of new situated abstractions, such as 'the number of trials large and the workings box fair causes the pie chart to appear even'.

In this chapter, we examine the children's subsequent expressions of meanings as they interact with the two-spinners and two-dice gadgets. Both these gadgets focus on the total of two independent events. In our privileged position, we are able to recognise that the mathematics underlying these two gadgets as more or less identical; the only distinction is that the two-spinners gadget is rendered more simple by dint of the fact that less combinations are involved. We might regard the meanings which evolve through interactions with these two gadgets as a special case of global meanings in the sense that children draw on previously constructed global meanings. Nevertheless, we will see that the process of meaning-making for these compound events makes its own special demands.

As in the previous chapter, the story of how the meaning-making unfolds is best told through the detailed analysis of one pair of children's interactions with the

Chance-Maker microworld. This case will outline most of the main issues which appear in many of the case studies. Again, Anne's and Rebecca's activity will serve as a clear illustration of the main issues. Using Anne and Rebecca as the main case allows a smooth continuation of the story as it unfolds from Chapter Eight. Anne's and Rebecca's case proved post hoc to be particularly effective as a vehicle for disseminating these issues but the full analysis of all eight case studies is available at the World Wide Web address: http://www.warwick.ac.uk/wie/staff/DP.htm. Out of the mass of data from the analysis of these case studies emerged the issues which Anne's and Rebecca's case neatly illustrates. There were of course some variations on these main issues and they are illustrated through a relatively brief analysis of two further cases studies.

9.2. ANNE AND REBECCA : A CASE OF MEANING-MAKING FOR COMPOUND EVENTS

Let us first consider the initial meanings for compound events as articulated in Anne's and Rebecca's pre-interviews and in their early interactions with the Chance-Maker microworld.

9.2.1. Initial Meanings for the Total of Two Dice

In their pre-interviews, both Rebecca and Anne declared that no total for two dice was harder or easier to obtain than any other. Their reasoning was however different. Anne intuits that the totals must be equally likely because the dice are (individually) fair and so the combination of them must be fair.

> "Because you can't estimate what number you'll get because they're all fair, both the numbers are fair."

Rebecca also saw the total of two dice as equiprobable but because the throws of the dice could not be controlled.

> "Cos it's random, you can't control which number it lands on."

These initial meanings for two-dice were confirmed early in their interactions with the two-spinner gadget.

> I ask, "If these were real spinners, and we can get any total between 2 and 6, do you think there is any total which is harder to get than the others; any total that's easier to get than any others?" Anne replies, "What it was in real life? *(I confirm.)* No." Rebecca adds, "There's a 50 / 50 chance of getting any total." I clarify, "So you think all the totals are equally easy or hard to get." They both say, "Yes."
>
> (6.10.2)

Even after using the two-spinner gadget, Anne and Rebecca continued to believe that the totals for two dice would be equally likely (see the opening paragraphs of the section, 'Anne and Rebecca use the two-dice gadget'). Below, I outline their interactions with the two-spinner and two-dice gadgets.

9.2.2. Outline of Anne's and Rebecca's Use of the Two-Spinner and Two-Dice Gadgets

Anne and Rebecca use the two-spinner gadget

Anne and Rebecca began by familiarising themselves with the two-spinners gadget (Figure 9.1).

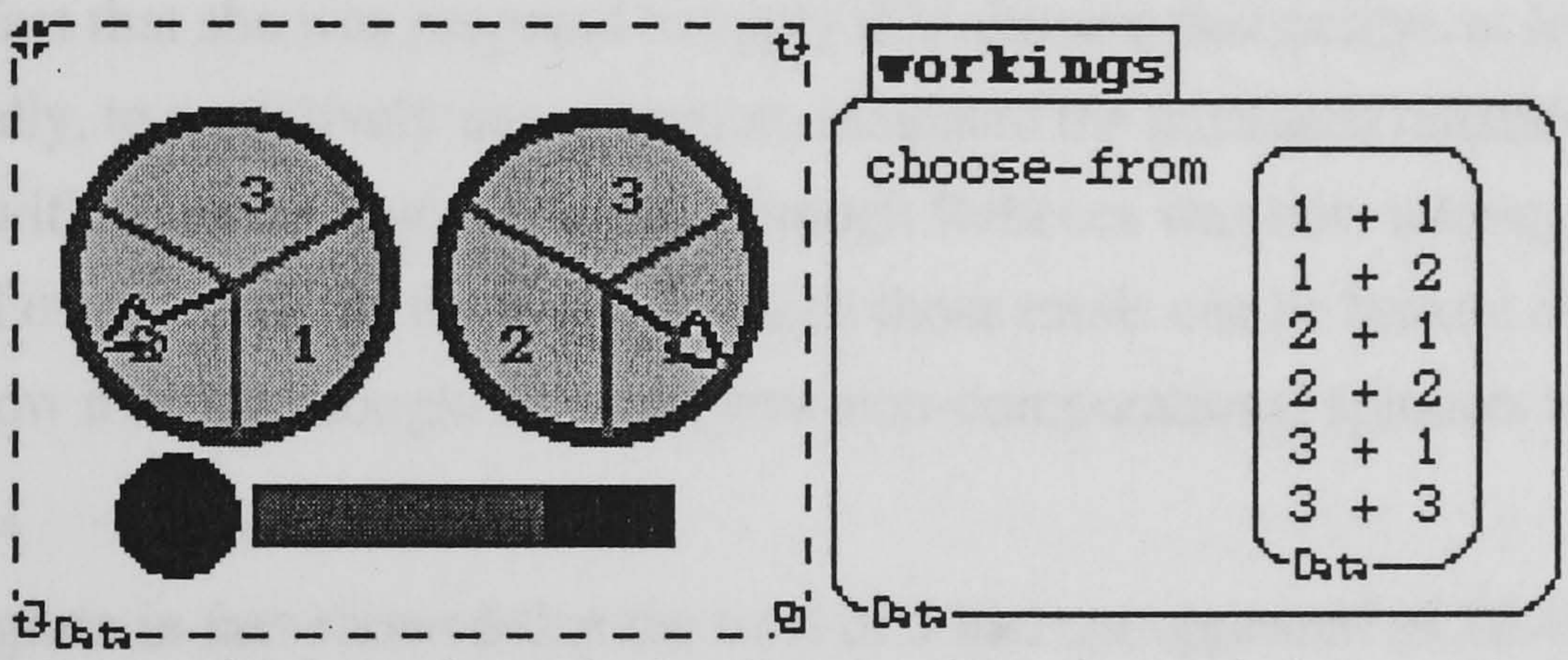

Fig. 9.1 : Anne and Rebecca began by familiarising themselves with the two-spinner gadget

I explained that the *total* of the two spinners was reported. I explained the workings including the role of the addition sign.

A pie chart indicated to the girls that perhaps too many 4's and 3's were appearing (6.10.3) (Figure 9.2).

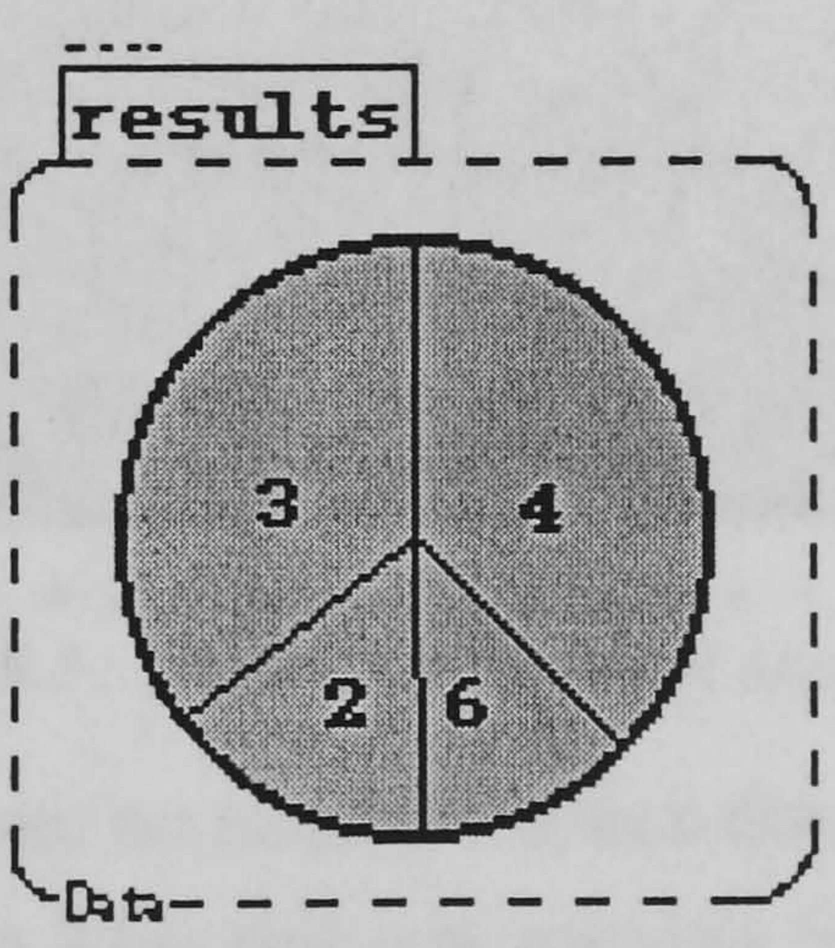

Fig. 9.2 : Too many 4's and 3's were appearing

Rebecca's situated abstraction from her previous work was cued as a possible explanation for the discrepancy.

> "Maybe if we do it more times, it might be more even. It was with the dice and the spinner."
>
> (6.10.3)

Certainly she was not prepared to accept the evidence of just 50 trials, and so they carried out 1000 trials. Whilst we waited for the pie chart to appear, it became clear that both girls believed that the 3's and 4's would in fact appear more often. This was not now based merely on the results of their early trials but on some analysis of the workings box. Rebecca explained:

> "Because most of the sums seem to come to either 3 or 4 .... The workings .... The second one down comes to 3 and then the third one down comes to 3. Then the next two down both come to 4."
>
> (6.10.3)

Rebecca was using her situated abstraction that the pie chart would not contain equal sectors even when the number of trials was large if the workings were not fair. The fact that she was prepared to apply this situated abstraction, at least conjecturally, to a relatively new situation, indicated the increased reliability of this meaning within her internal resources. Though Rebecca was now relating the likelihood of each total to the ways in which those totals can be broken down, we do not know that she thought this was how non-computational spinners would behave.

The pictogram in fact showed that the total of 5 had not appeared (6.10.4) (Figure 9.3).

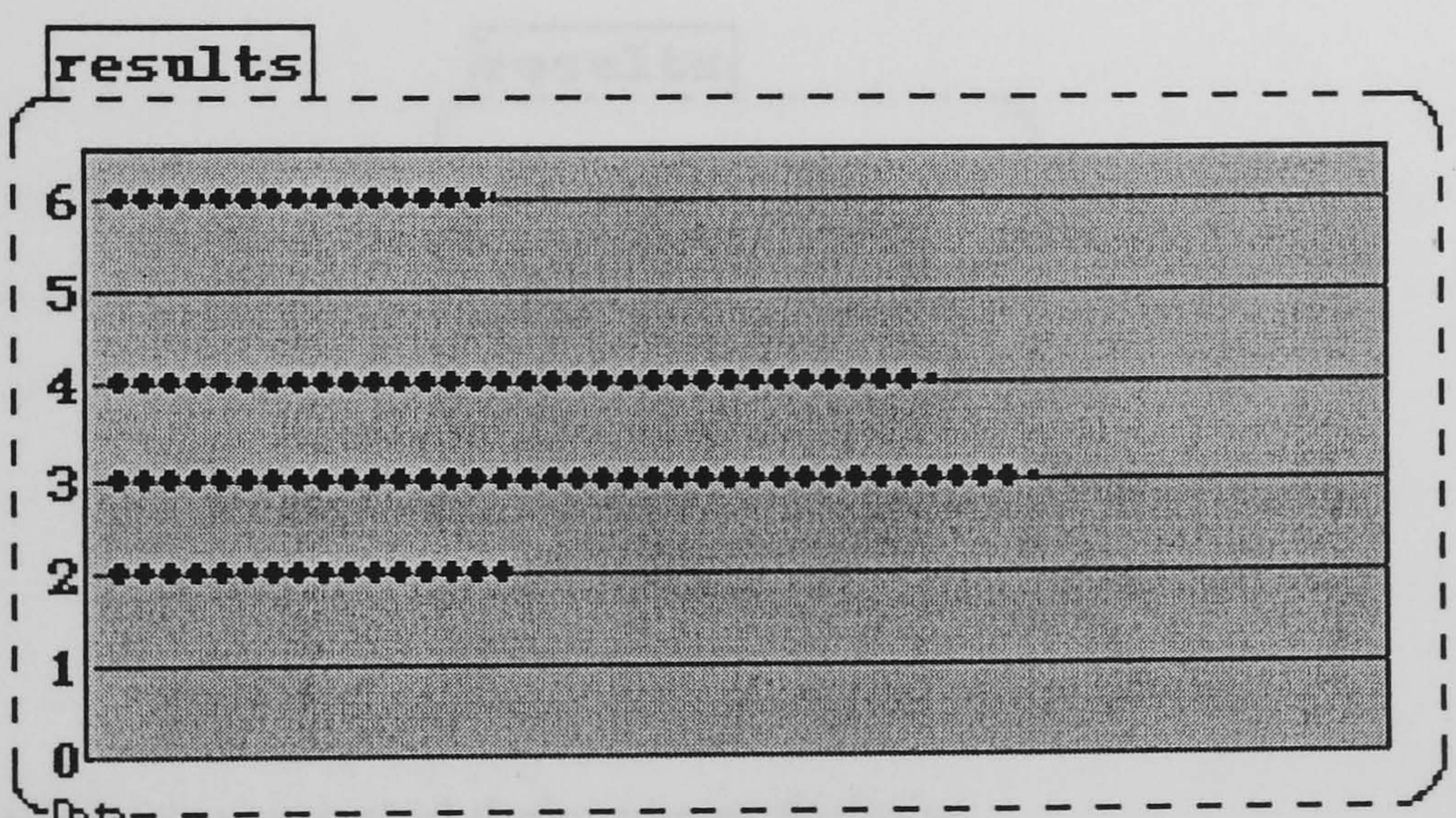

Fig. 9.3 : The total of 5 had not appeared

Anne suggested that there were no numbers which made 5, but Rebecca pointed out that you could have a 2 and a 3 but this was missing from the workings. They edited the workings to read

choose-from [1+1 1+2 2+1 2+2 3+1 2+3 3+2 3+3] (Figure 9.4).

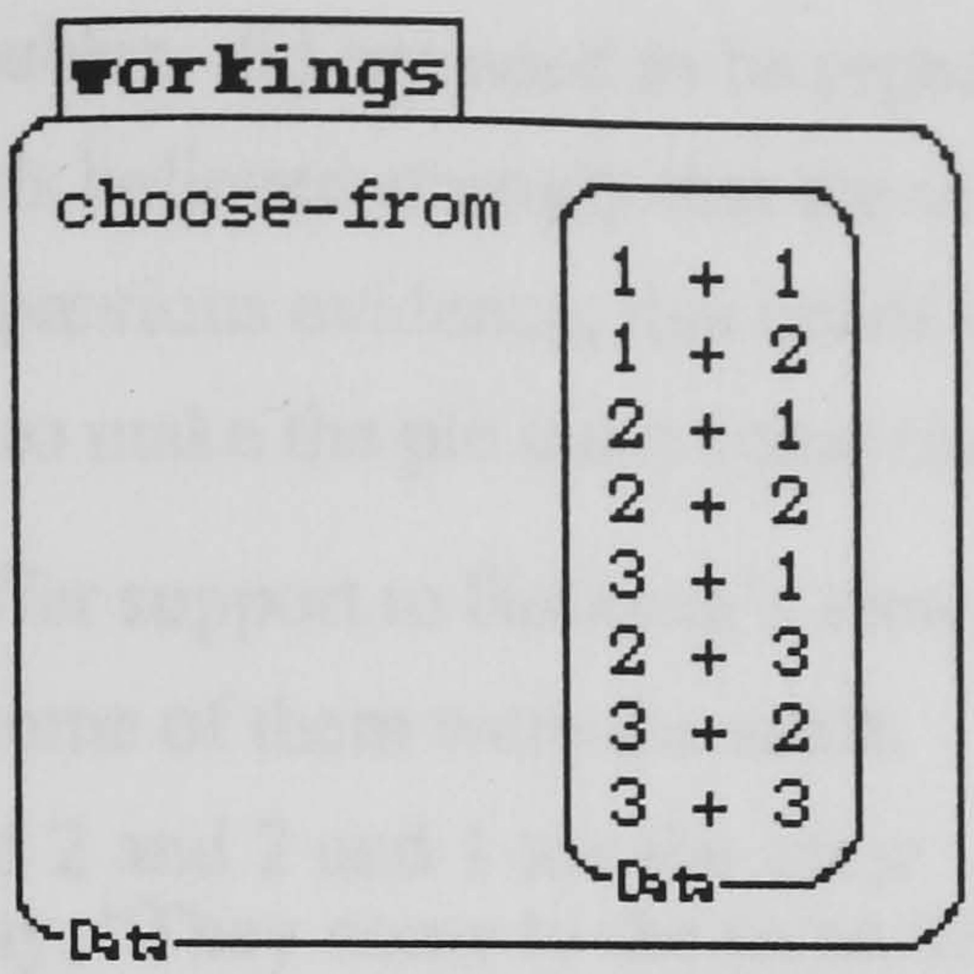

Fig. 9.4 : They edited the workings to include 2+3 and 3+2

Rebecca explained that 2+3 was needed as well as 3+2:

> "because the first number is representing the first spinner, so you have to have it both ways."
>
> (6.10.4)

Rebecca was clear that the doubles did not need to be repeated. The girls however missed the outcome, 1+3. They repeated 1000 new trials with these new workings (6.10.5). Anne predicted that the pie chart would be even, (presumably on the basis that the spinners were fair). The pie chart showed unequal sectors with least 2's (Figure 9.5).

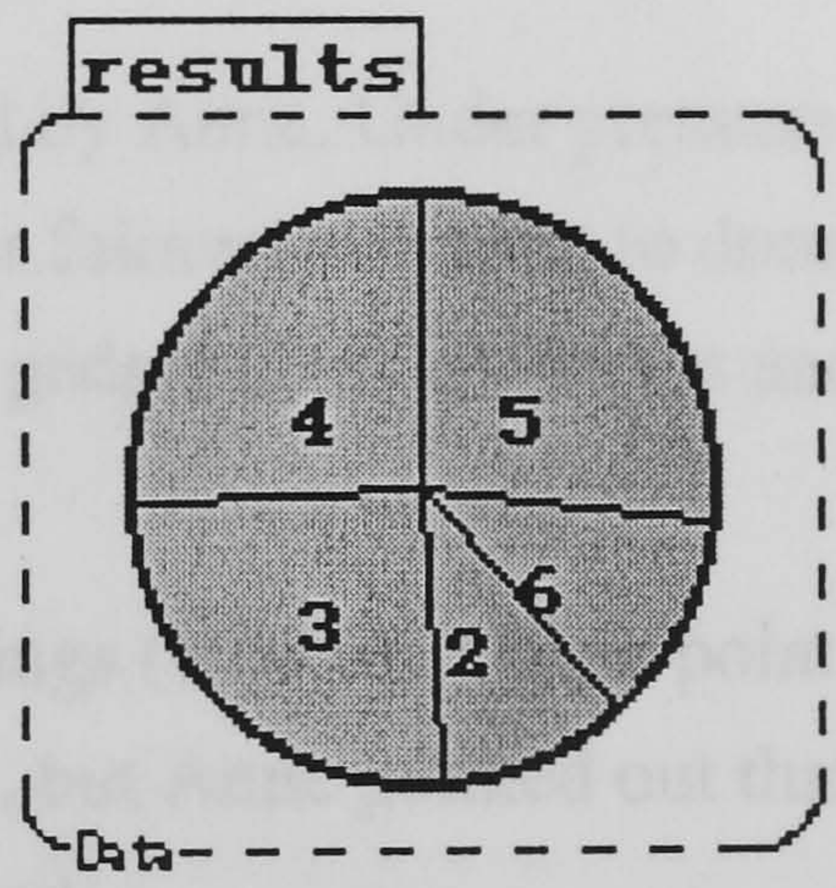

Fig. 9.5 : The pie chart showed unequal sectors with least 2's

The next interchange revealed their current thinking.

> Anne says, "Maybe we should put 1 add 1 in again." Rebecca objects, "No, because it's already been done so it would just be the same." Anne argues, "Yes, because some of them have got the same again." Rebecca agrees "Yes, maybe we should actually put some of them in again, because then there's more chance of them coming out more even."
>
> (6.10.5)

Anne saw the insertion of another 1+1 as a way of equalising the sectors of the pie chart. Rebecca was torn between her meaning that the different totals should be equally likely (because they could not be controlled) and her analysis that the doubles, unlike the non-doubles, did not need to be repeated in the workings. It was now clear that both girls believed strongly that the different totals should be equally likely, confirming previous evidence; this belief was so strong that they sought to fix the workings to make the pie chart come out even.

I decided to intervene to offer support to Rebecca's view. I asked Anne what she meant when she said that some of them were the same.

> Anne: "Well, 1 and 2 and 2 and 1 are the same .... they come to the same number." I say, "They come to the same total, but are they the same as far as the spinners are concerned?" Rebecca explains, "No they are not. Because, the second one down, that number *(pointing to the 1 of 1+2 in the workings box)* refers to that spinner *(pointing to the first spinner)*, and that number *(pointing to the 2 of 1+2)* refers to that spinner *(pointing to the second spinner)*. So, say, if that one *(the first spinner)* lands on 1 and that one *(the second spinner)* lands on 2, it would be three. And if that one *(the second spinner)* lands on 1 and that one *(the first spinner)* lands on 2, it would be three as well." Anne says, "Exactly .... I think we should add that one *(pointing to the 1+1)* and that one *(pointing to the 2+2)* again because then we get more of a chance of getting them." I say, "But then you would be putting in 1 plus 1 twice." Anne responds, "Yes, because 2 doesn't come up as much, does it?" Rebecca agrees, "So maybe if we do that."
>
> (6.10.5)

My intervention was rejected by Anne. Under pressure from Anne, and despite my support, Rebecca allowed her fairness meaning to dominate. There was an impressive need to make the gadget give fair results and so behave as they felt real spinners would.

They began to edit the workings (6.10.6). At one point Rebecca was about to add another 2+2 as well as a 1+1, but Anne pointed out that they already had lots of 4's. Finally, the workings read:
choose-from [1+1 1+1 1+2 2+1 2+2 3+1 2+3 3+2 3+3 3+3] (Figure 9.6).

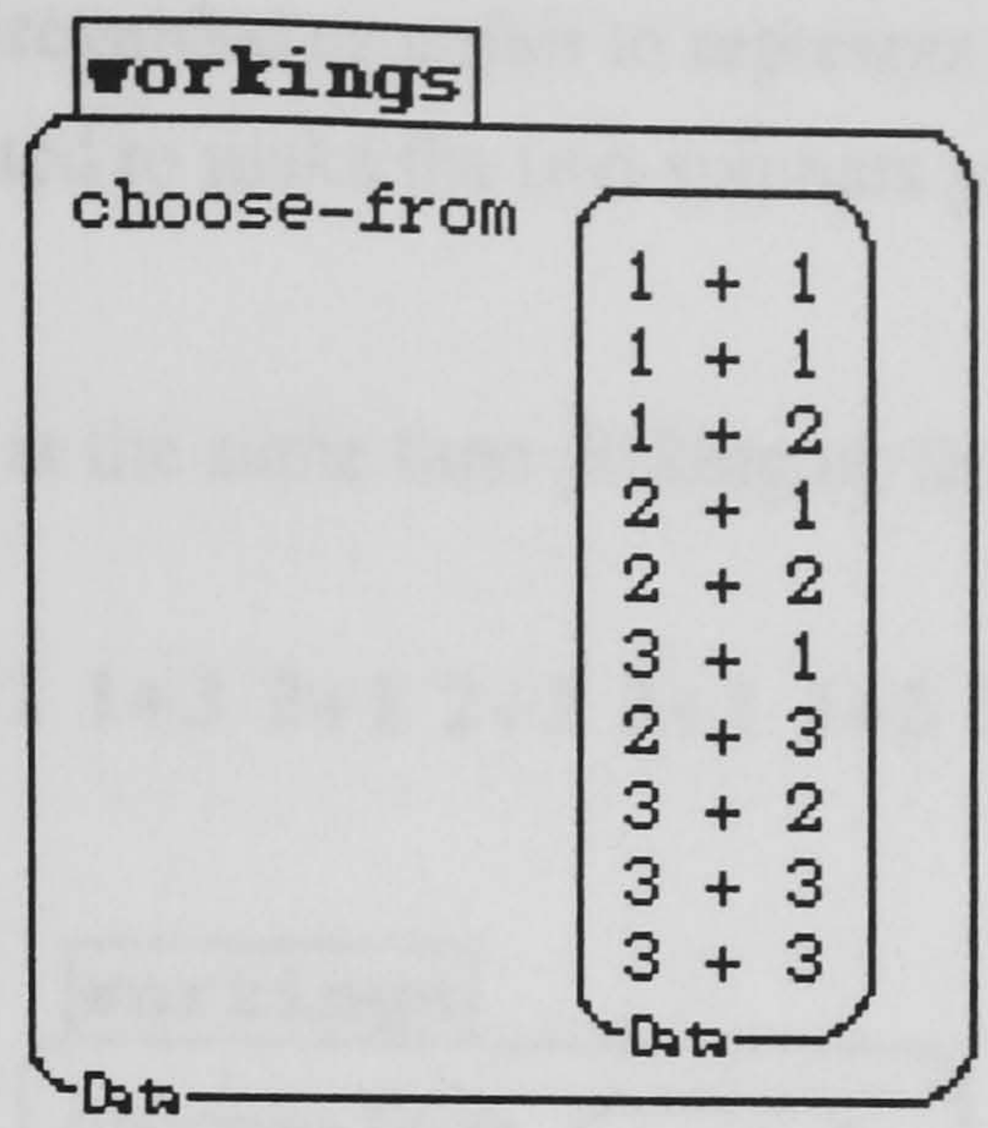

Fig. 9.6 : The workings were edited to make the pie chart 'fair'

The inconsistency of adding an extra 1+1 and not an extra 2+2 to the workings was not a concern for Anne since her aim was simply to equalise the likelihoods of the different sectors, and hence make the gadget work properly in her view. After 1000 new trials, the pie chart was fairly even (6.10.7) (Figure 9.7).

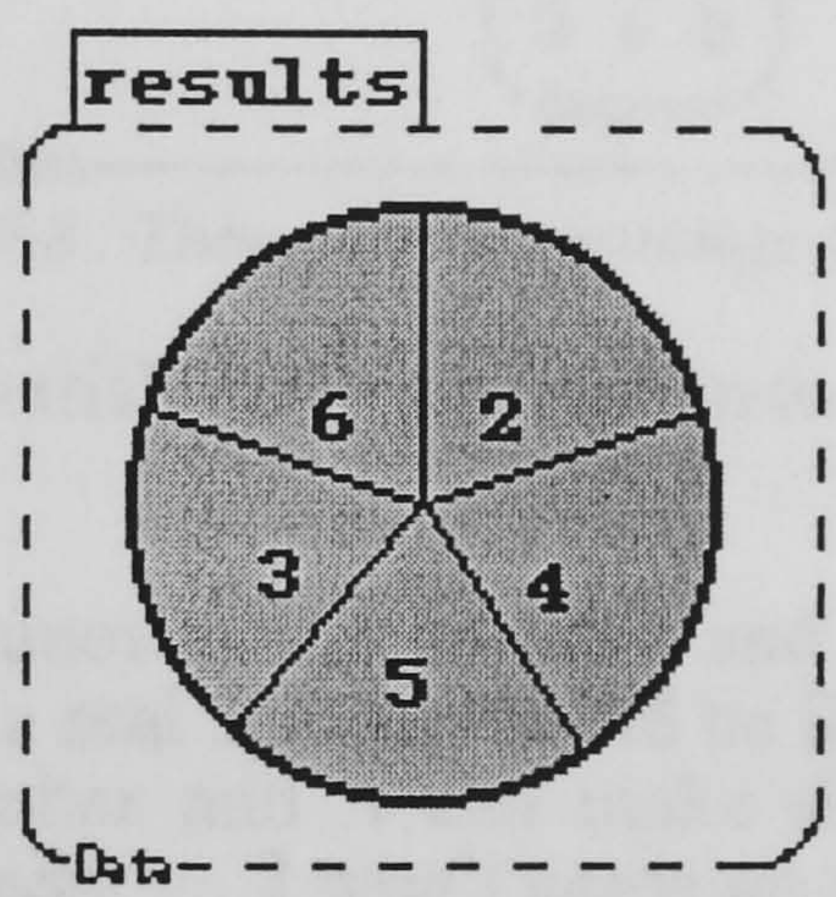

Fig. 9.7 : The pie chart was fairly even

I needed to re-problematise the situation so that Anne and Rebecca might question their solution.

> I point out, "What we don't know for sure is whether that is how real spinners would behave. I think what you need to try and do is justify why you should have 1 plus 1 in there twice over." Rebecca replies, "Because everything else has two ways of coming except maybe 2 plus 2." I ask, "But in reality, does 1 plus 1 have two different ways of coming?" Rebecca argues, "I think it is more fair because the pie chart looks roughly even and before there were barely any 2's and barely any 4's ..." I respond, "I think you have certainly made it more fair. What I am not convinced about is that you have made it more like real spinners would be .... maybe with real spinners that would not be the case." Anne: "Oh, yes, mmm." I say (6.10.8), "You see, I am not sure you are being fair by putting 1+1 in twice." She continues, "We don't want it to be even. We want it to work like a real spinner."
>
> (6.10.7)

My intervention aimed to suggest an alternative way of thinking about fairness by suggesting that it might be regarded as unfair to represent the same outcome twice over, especially as we wanted to make the two-spinners gadget behave like real spinners.

They edited the workings, at the same time picking up the previously missing 1+3. The workings now read:

choose-from [1+1 1+2 1+3 2+1 2+2 3+1 2+3 3+2 3+3] (Figure 9.8).

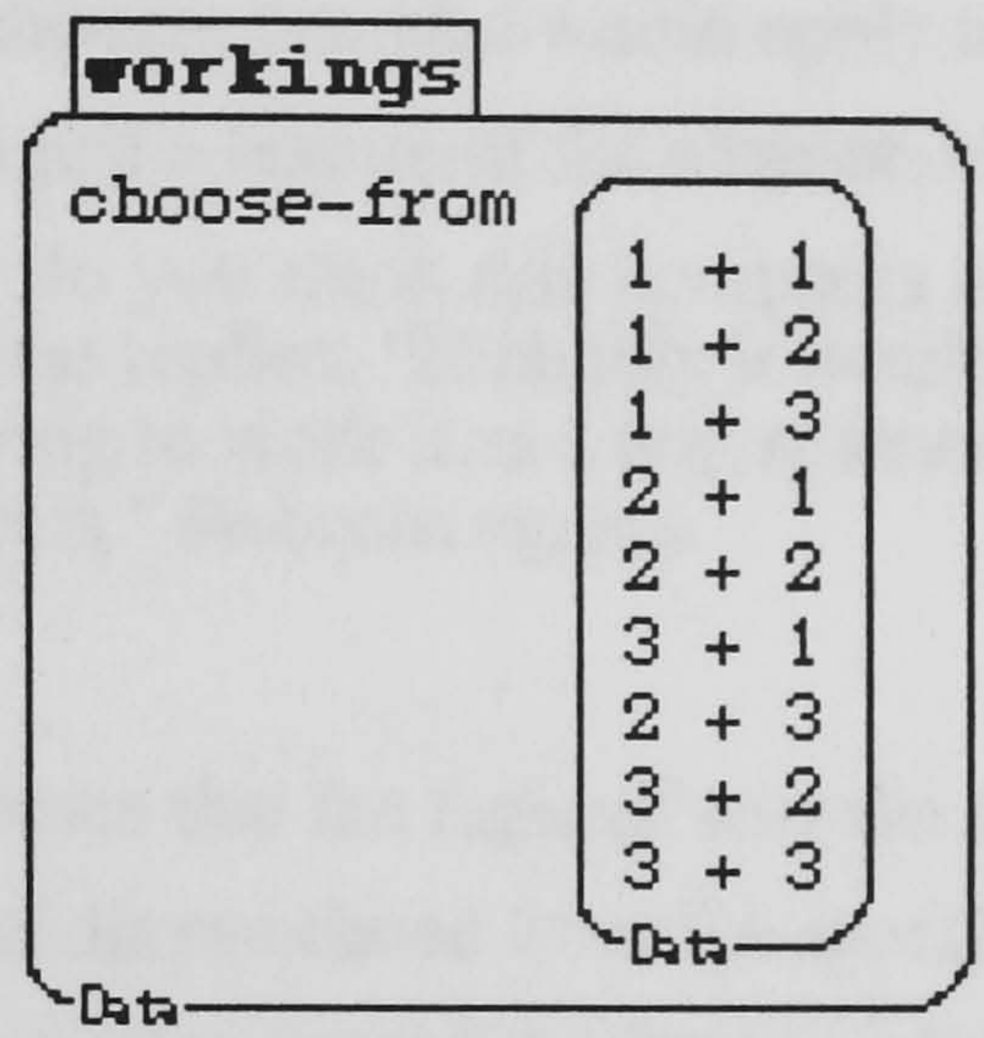

Fig. 9.8 : They made the workings 'fair'

The girls generated 1000 new trials and I asked them to predict what the pie chart would look like.

> Anne replies, "A bit uneven .... because 1 and 1 has only got once, because that is what a real spinner would be like. And the rest has got like double number and it can make different numbers. " Rebecca: "I think maybe .... 2 won't come up as much and 6." The pie chart shows most 4's and least 2's and 6's, with slightly less 6's than 2's.
>
> (6.10.8)

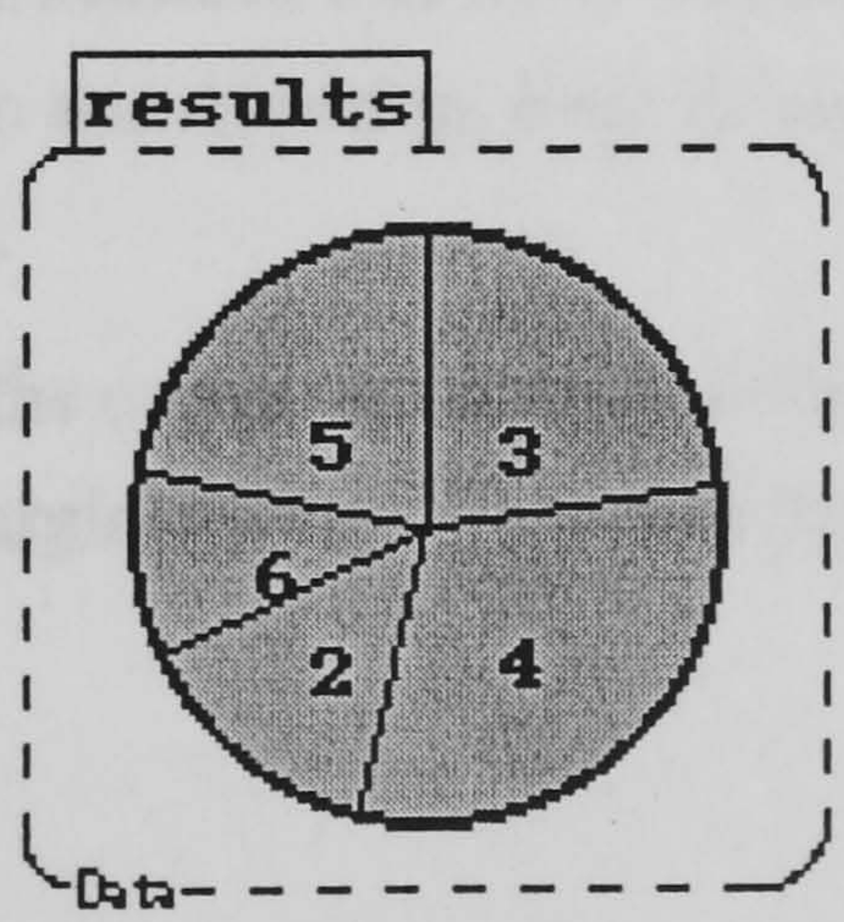

Fig. 9.9 : The pie chart shows most 4's and least 2's and 6's, with slightly less 6's than 2's

For the first time, Anne and Rebecca suggested that the two-spinner gadget should not generate an even-looking pie chart. I looked for an explanation from them for the size of the 4 sector in this pie chart (6.10.9). Rebecca responded:

> "I've seen it twice .... It's written there *(in the workings)* three times."
>
> (6.10.9)

They continued by counting how many times each of the other totals was represented in the workings. It was clear that Anne and Rebecca had a new situated abstraction, which could be schematised as: 'The more often a total is represented in the workings box, the larger will be its sector in the pie chart'.

I wondered whether they thought this idea would apply to non-computational spinners or whether it was just a feature of the Chance-Maker microworld.

> I ask, "Now, how do you think this compares to doing it with two real spinners?" Anne replies, "Probably it would be about the same because we are trying to work it as a real spinner, and we've got the same sort of numbers." Rebecca agrees.
>
> (6.10.9)

There was then some evidence that the logic of why the pie chart was uneven was sufficiently compelling and disassociated from the specifics of the microworld context for them to be prepared to extend the domain of the situated abstraction to non-computational spinners. Further questions from me revealed that they thought that the 4 was the easiest total to obtain from two spinners, but that the 6 was a little harder to obtain than the 2.

> Rebecca adds, "But 6 seems harder because it is smaller ...." Anne interjects, "Yes, much smaller." Rebecca continues, "Less of 6."
>
> (6.10.9)

The new situated abstraction then was not so robust that it could not be dominated by other factors. Both girls articulated that the 6 was less likely than the 2 because the results seemed to point in that direction, even though the workings suggested that they were equally likely.

I suggested that they repeat the experiment again (6.10.10). This time the pie chart showed that 2's occurred marginally less often than the 6's.

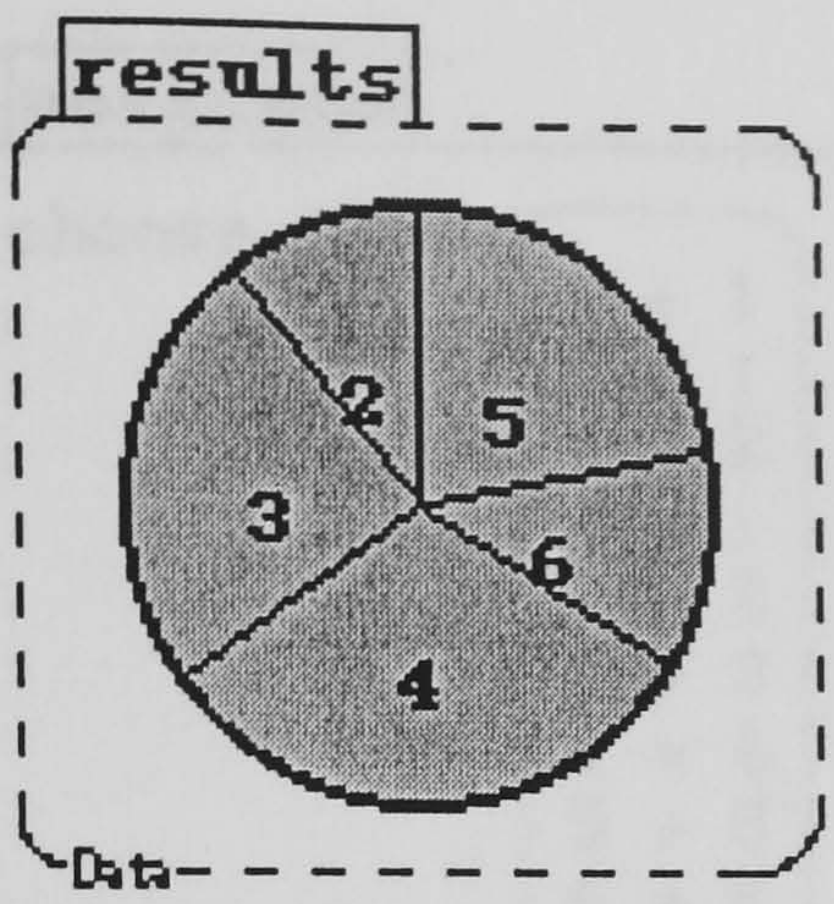

Fig. 9.10 : The pie chart showed that 2's occurred marginally less often than the 6's

Rebecca and Anne agreed that the 6 and the 2 were in fact equally hard to get.

Now that the evidence from the results was contradictory, Anne and Rebecca were prepared to rely more heavily on their situated abstraction. At this point, attention switched to the two-dice gadget.

Anne and Rebecca use the two-dice gadget

Anne and Rebecca began to use the two-dice gadget (Figure 9.11).

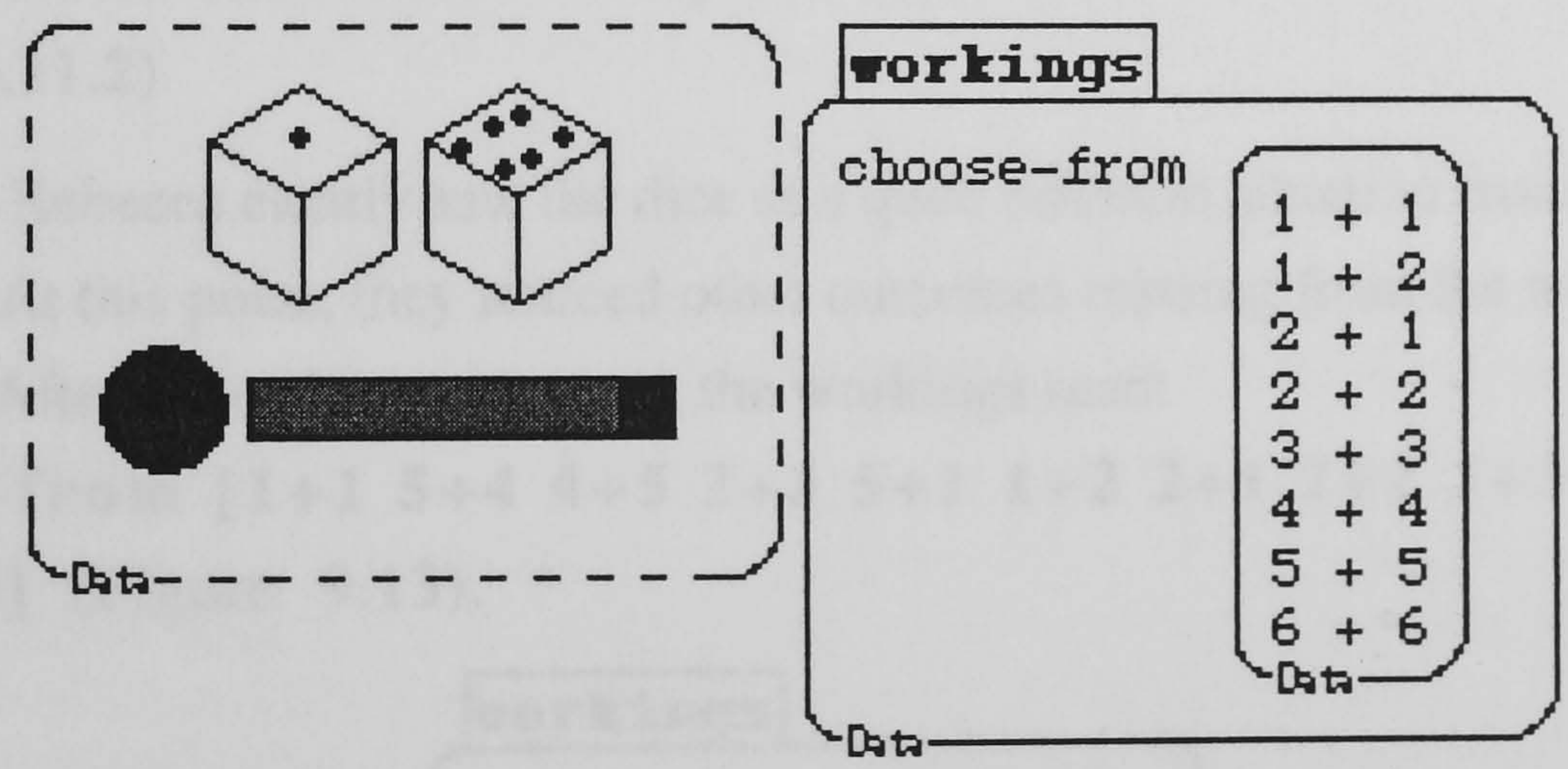

Fig. 9.11 : Anne and Rebecca began to use the two-dice gadget

It was clear from their early discussions that they believed that there was missing data in the workings (6.11.1). In particular, Rebecca observed:

> "There's lots of different ways .... There's probably more ways of making 6 .... ." Anne points out, "Ah, 5 add 1 is missing to make 6."
>
> (6.11.1)

They inserted 5+1 below 1+1 in the workings (Figure 9.12).

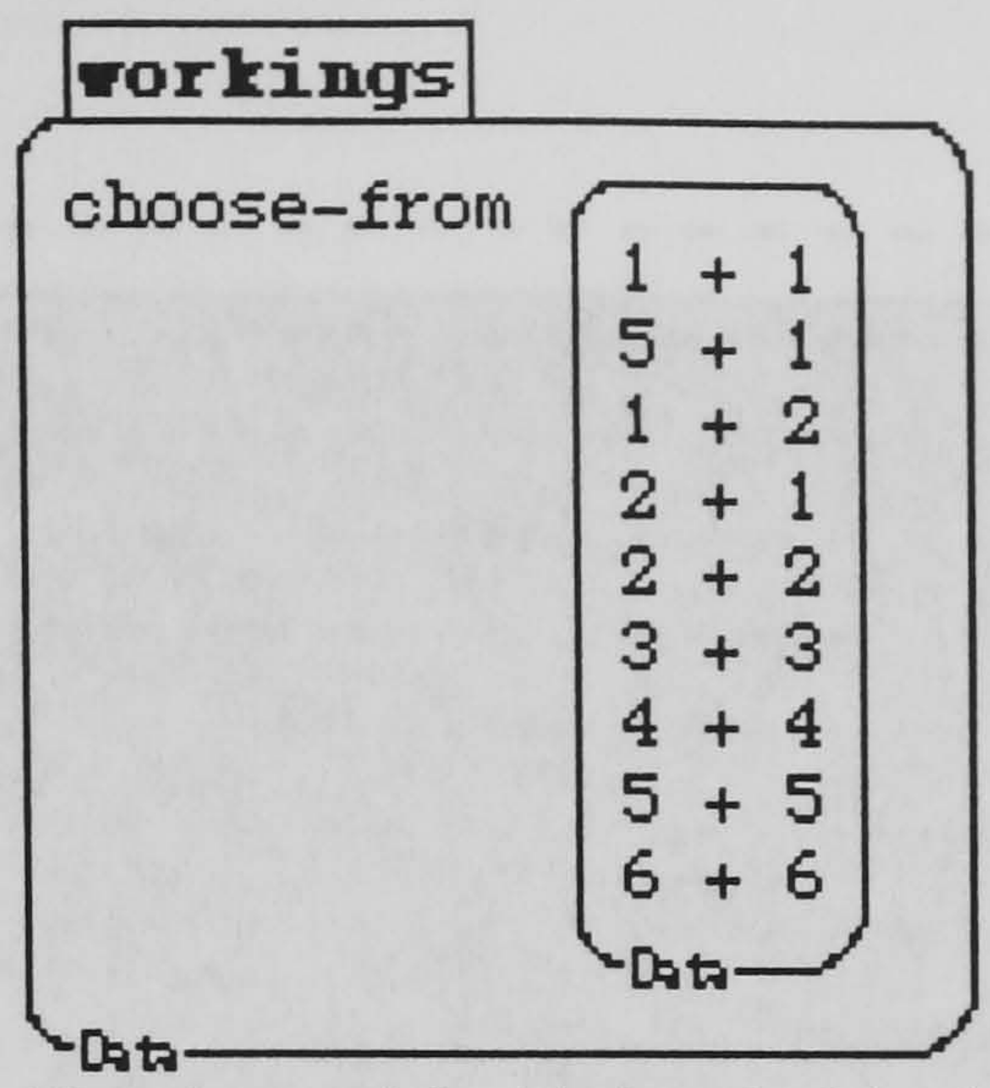

Fig. 9.12 : They inserted 5+1 below 1+1 in the workings

At this point I wished to find out whether their thinking about the totals of two dice had changed in the light of their situated abstractions from the two-spinners gadget.

> I ask, "If we were shaking two real dice, do you think all the totals you could get are just as easy, just as hard, or do you think some totals are easier than others, harder than others?" "50 / 50 chance of getting them." Anne agrees. I clarify, "So you think they are all about the same chance?" They both say, "Yes."
>
> (6.11.2)

Anne and Rebecca clearly saw the dice as a quite different situation from the spinners. At this point, they noticed other outcomes missing from the workings box (6.11.3). After several amendments, the workings read:

choose-from [1+1 5+4 4+5 2+3 5+1 1+2 2+1 2+2 3+3 4+4 5+5 6+6 3+2] (Figure 9.13).

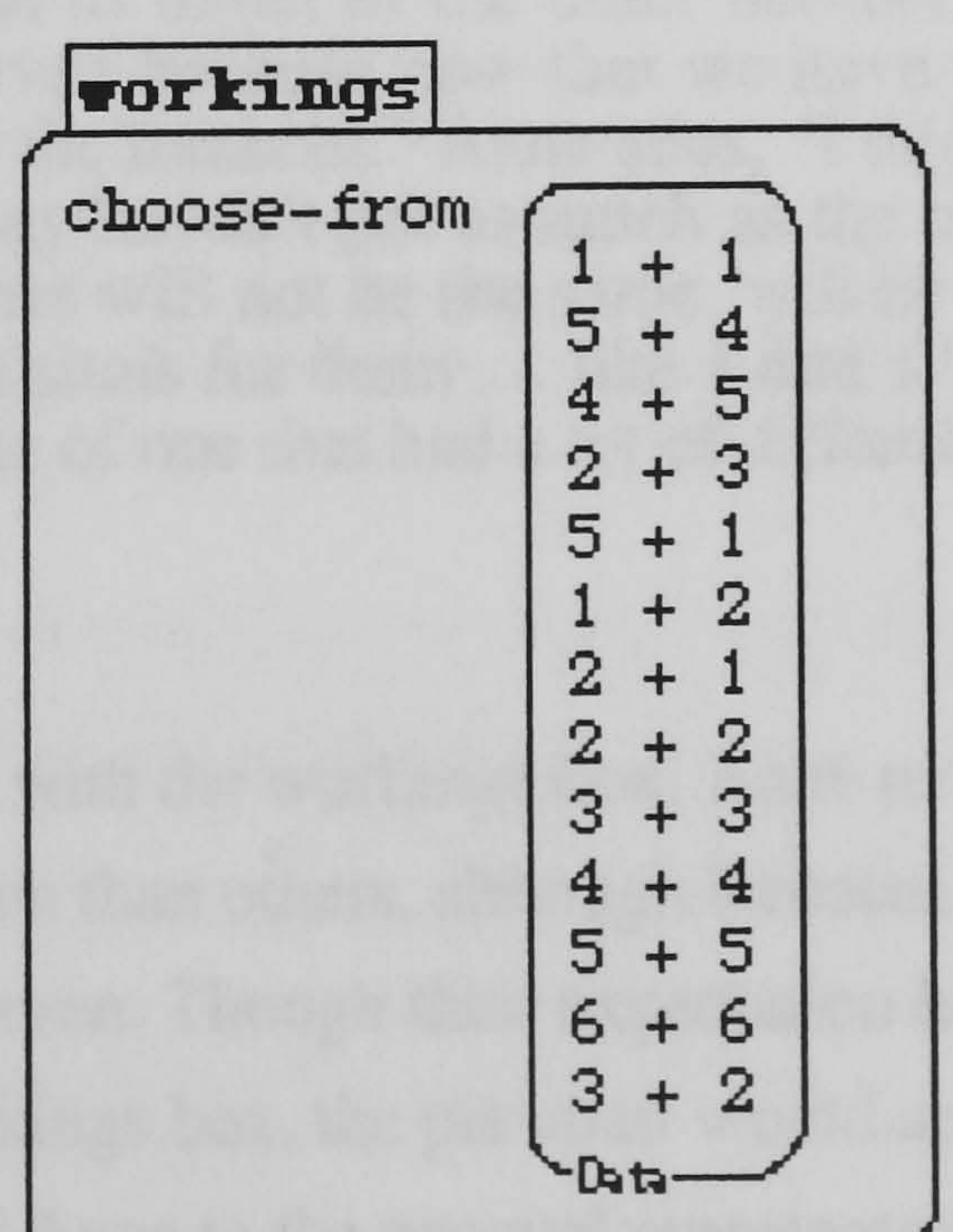

Fig. 9.13 : They edited the workings to include some missing outcomes

They repeated 1000 new trials to help them to discover which other outcomes were missing. The pictogram showed that there were no 7's nor 11's[27] (6.11.5) (Figure

9.14).

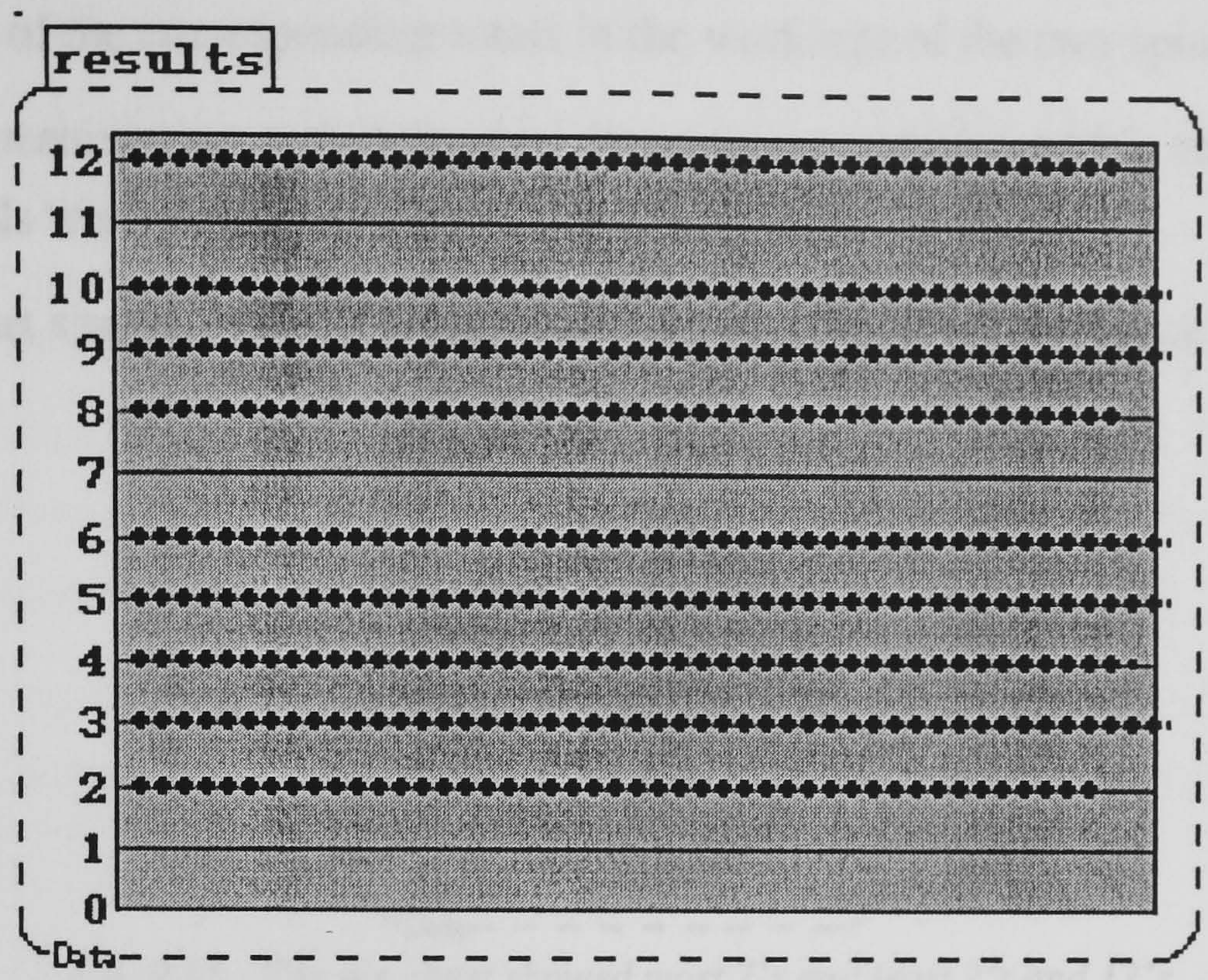

Fig. 9.14 : The pictogram showed that there were no 7's and 11's

They inserted 6+5 and 5+6 at the bottom of the workings box. They then appended 5+2, 2+5, 3+4, 4+3, 6+1 and 1+6. I suggested that they check the other totals systematically. They needed only a little support from me to work through totals 2 to 12 until eventually they had included all 36 combinations.

They repeated 1000 new trials (6.11.6). I asked what they thought the pie chart would look like.

> Anne replies, "Fairly even? ....Some of the numbers might not be because there's not as much as the other number." Rebecca says, "Maybe roughly even because now that we have got all the sums. I'm not too sure at the moment." Anne adds, "I think some will be a bit less because they haven't got as much as the others .... because some of the numbers will not be the same, will be less, because we didn't find enough sums for them .... like 1 add 1." I ask, "Can you give me an example of one that had a lot of different ways of getting it." Anne: "7."
>
> (6.11.6)

Through their interactions with the workings box, Anne recognised that some totals were represented more often than others, although Rebecca still thought the pie chart might turn out to be even. Though their expectation had been that, by inserting the extra data into the workings box, the pie chart would appear more even, the editing process had alerted Anne to the unequal representation of different totals. Their previous experience with the two-spinners gadget offered the possibility that the pie chart might be uneven as a result. There were then two internal resources available to Anne and Rebecca:

• the situated abstraction that the sectors in the pie chart were determined by the frequencies of the corresponding totals in the workings of the two-spinners gadget,

• the local meaning that each individual dice was unsteerable and fair and so the various totals were equally likely.

The pie chart showed most 7's and least 2's and 12's (6.11.7) (Figure 9.15).

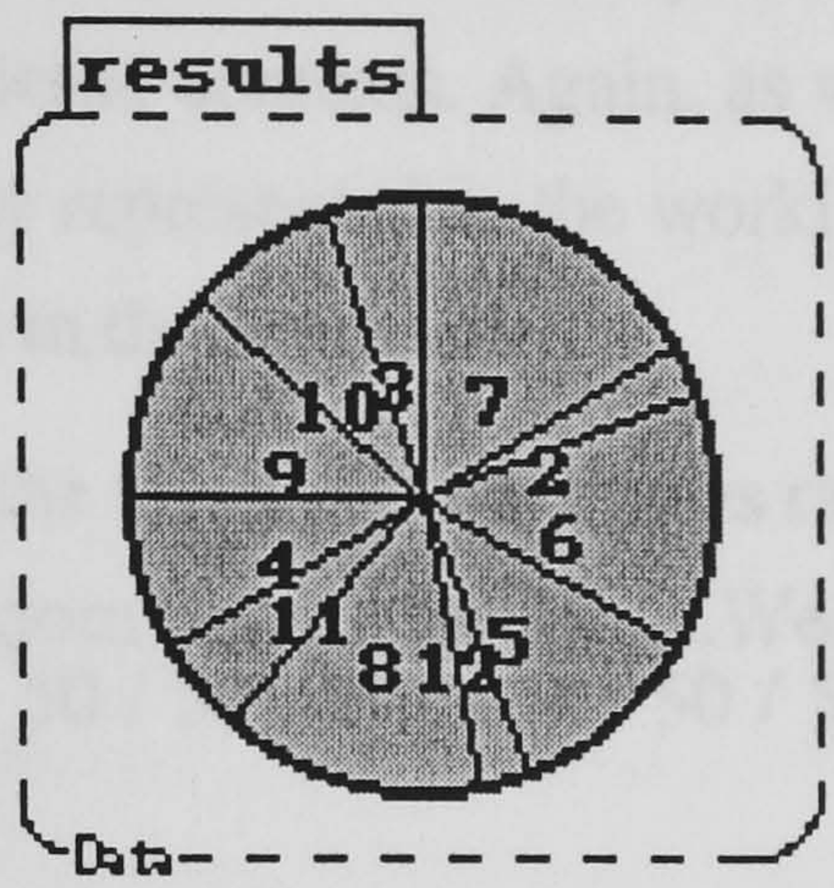

Fig. 9.15 : The pie chart showed most 7's and least 2's and 12's

Rebecca's first reaction was that there might still be some missing outcomes in the workings box. I reassured her that this was not the case (in hindsight, an intervention which might have prevented further illumination of Rebecca's meanings). Rebecca conjectured:

> "Ah, I bet there are various ways of making a number. There can be more ways of making one number than there can be of another."
>
> (6.11.7)

By inspecting the workings box, the girls identified that the 12's, 2's and 3's did not have many ways whereas the 7 had the most. I asked why then the pie chart had that appearance (6.11.8). Anne replied:

> "Because some of the sums we put as more. Like 2, we could only get about one of them so that's why it came out a bit like that."
>
> (6.11.8)

I wondered whether they would be prepared to extend the domain of this situated abstraction to non-computational contexts.

> I ask, "So when you are at home, and you are playing dice with your brother, Rebecca, and you've got two dice, and you say to him, 'I bet you 10p what the total's going to be.' You go for 7. What are you going to tell him to go for?" Rebecca: "2 .... no, sorry 12 .... 12 is smaller than 2." I ask, "So, do you now think that all the totals on two dice are just as easy or just as hard as each other? Or do you think some totals are easier than others?" They both say that some totals are easier to get.
>
> (6.11.8)

The leading nature of my question may well have influenced Rebecca's response, so there is a need to be cautious in concluding that Rebecca may have been prepared to extend to non-computational contexts her situated abstractions that the totals were not equally likely, and that how likely they were depended upon how often they were represented in the workings. This situated abstraction was as true of two dice as it was of two spinners and seemed to extend beyond the Chance-Maker microworld to non-computational contexts. Again, as with two-spinners, where the least likely totals were equally represented in the workings box, she gave a casting vote to the actual frequencies in the results.

Indeed, Rebecca did not see the totals of two-dice as random any more (6.11.9).

> "It doesn't seem random now, does it? ....Well, not really random because they are not 50 / 50 .... It's not 50 / 50 because you don't get the same number."
>
> (6.11.9)

9.2.3. A Trace of Anne's and Rebecca's Use of the Two-Spinner and Two-Dice Gadgets

We can track the process by which the two girls constructed new meanings for compound events as encountered during their interactions with the two-spinners and two-dice gadgets. In Table 9.1, I refer to conjectures, initial meanings and situated abstractions, as defined in Chapter Eight. The table below also indicates the critical interventions.

Para	Conjectures (C), Initial Meanings (IM) and Situated Abstractions (SA)	Critical Interventions
6.10.2 to 6.10.3	Two spinners in everyday contexts are fair and so the totals are equally likely (IM) The number of ways the totals are represented in the workings controls the size of the sectors in the pie chart (provided the number of trials is large) (SA)	

6.10.4 to 6.10.7	*Activity: A & R do 1000 new trials. The pictogram shows that 5 is missing. They edit the workings to include 2+3 and 3+2. They repeat a new 1000 trials. The pie chart is uneven. A & R debate whether the doubles should be written twice into the workings. Anne's argument, that putting 1+1 in again, but not 2+2, would equalise the pie chart, persuades Rebecca.. They edit the workings to include another 1+1. They repeat 1000 new trials. The pie chart looks even.*	
6.10.7		You have put 1+1 in twice but 2+2 only once. Is it fair to put 1+1 into the workings twice?
6.10.7 to 6.10.8	*Activity : A & R edit the workings to exclude one of the 1+1's and they include 1+3. They repeat 1000 new trials and the pie chart shows most 4's and least 2's and 6's.*	
6.10.9 to 6.10.10	4 is easier to get than the other totals because there are more ways of making 4 (SA) 6 and 2 are harder to get because they have less ways in the workings, but 6 is even harder because the sector in the pie chart is smaller than that for 2 (SA)	Repeat the experiment.
6.10.10	*Activity: A & R repeat 1000 new trials. The pie chart is similar except that the 6's are more frequent than the 2's.*	
6.10.11	6 is the same as 2 (SA) 4 is easier to get than 2 for non-computational spinners as well as for computer gadgets (SA)	
6.11.1	*Activity: A & R begin to use the two-dice gadget. They think that some totals are missing and begin to insert data into the workings..*	
6.11.2	A & R: Totals for two everyday dice are equally hard or easy (IM)	
6.11.3 to 6.11.6	*Activity: A & R make further additions to the workings and decide to repeat 1000 new trials to help identify which other ones are missing. The pictogram shows 7's and 11's are missing. They add combinations that make 11 and 7. They then continue systematically until they have all 36 combinations..*	

6.11.6	A: Perhaps the pie chart will not be even because some totals appear more often in the workings (C)	
6.11.6 to 6.11.7	*Activity: A & R repeat 1000 new trials.. The pie chart shows most 7's and least 2's and 12's.*	
6.11.8	Some totals are more likely than others, even for non-computational dice (SA) R: Two dice are not really random because the totals are not 50/50. (SA)	

Table 9.1 : A trace of Anne's and Rebecca's construction of meanings for compound events

9.2.4. Discussion of Anne's and Rebecca's Use of the Two-Spinner and Two-Dice Gadgets

The case of Anne and Rebecca illustrates the co-ordination of meanings for fairness. This co-ordinating process can be set out in four phases.

Expression of two meanings for fairness

The girls began the episode by expressing two meanings for fairness:

- The total of two spinners should be fair (and so equiprobable) because the spinners are symmetrical, unpredictable and unsteerable,
- The totals should be fair if we can include all the possible outcomes in the workings box.

The first type of fairness was articulated through statements such as:

> "You can't estimate what number you'll get because they're all fair, both the numbers are fair."
>
> "Cos it's random, you can't control which number it lands on."
>
> "There's a 50 / 50 chance of getting any total."

Anne and Rebecca expected that a large number of trials would make the pie chart even:

> "Maybe if we do it more times, it might be more even."

The pie chart though contradicted this expectation. When the pictogram depicted missing totals, the girls were able to argue that some totals were missing and so the unfairness in the pie chart was connected with the incompleteness of the workings.

Through the omission of some totals, the default value of the workings box had provoked engagement. Rebecca especially was confident that, by finding all the possible outcomes and entering them into the workings box, the two-spinners gadget would generate an even pie chart for large numbers of trials.

Making the pie chart fair

The second phase of their activity involved finding all the outcomes and entering them into the workings box. They were helped by the structure of the workings box and the appearance of the two-spinners gadget itself. They were able to make connections between the entries, such as 1+3, in the workings box, and the resting places of the arms of each spinner, for example, on a 1 for the first spinner and a 3 for the second.

The process of breaking the spinner down into the range of simple events was not particularly problematic, though they did accidentally omit one outcome. When the pie chart refused to come out even, they came to the conclusion that it was necessary for them to amend the workings further, increasing those totals which were under-represented and decreasing those which appeared too often according to the uneven pie chart. This strategy was consistent with thinking which regarded the equiprobability of the outcomes as the goal to be achieved. The former meaning of fairness was proving stronger than the latter.

Making the workings fair

Anne and Rebecca might have settled for the resulting even-looking pie chart had I not intervened. I intervened to raise the question of whether it was fair to include some outcomes more than once and not others.

This intervention spurred a new phase of activity, characterised by the intention to make the workings fair, not in the sense that each total was equally represented, but in the sense that each simple event was represented just once in the workings.

The uneven pie chart, together with their new construction of fairness, persuaded the girls that the totals for the two-spinners gadget should not be equally likely and that this would also be true of non-computational spinners.

> "Probably it would be about the same because we are trying to work it as a real spinner, and we've got the same sort of numbers."

This new meaning for fairness had not replaced other meanings. Note for example how Anne and Rebecca expressed the thought that a 6 was harder than a 2 because it's sector on the pie chart was smaller. Though this meaning was retracted when the experiment was replicated, Anne was prepared to make sense of the difference between 6 and 2 in terms of the fairness of the pie chart rather than the workings, demonstrating that this notion was still available. We also saw how Rebecca concluded at the very end of the session that the total of two dice did not any more seem to be random because the various outcomes were not equally likely. I interpret this as evidence that Rebecca has not replaced the notion of fairness based on symmetry by a new meaning for fairness but has enriched her earlier meanings with new variations; the earlier meanings can still be cued given the appropriate circumstances.

Re-constructing fairness for the two-dice gadget

The final phase of this episode involved the use of the two-dice gadget. We observed Anne and Rebecca repeating many of the above phases, though in a contracted manner. Despite their work with the two-spinners gadget, Anne and Rebecca stuck to their earlier meaning for the totals of two dice, expecting them to be equally easy or hard to obtain. For them, the two-dice gadget was a new situation; the two-spinners gadget appeared to be irrelevant.

This is not to say that the earlier experience had no influence. Anne and Rebecca began the search for other combinations more quickly and were not satisfied until all 36 had been found. The important example of learning was that when the pie chart was uneven for the two-dice gadget, they did not try to redress the balance by amending the workings. Instead they reviewed how often each total appeared in the workings and concluded that their original thinking was wrong, and that 7's appeared more often than other totals, whilst totals like 2 and 12 were relatively rare. As before though, they did make a distinction between the extreme cases of 2 and 12 on the basis of which appeared most often in the pie chart.

Although Anne and Rebecca did not make the transition from two-spinners to two-dice automatically, once they had constructed a meaning for the unevenness of the totals of two-dice, they were prepared to consider this as true of non-computational dice. The move from the Chance-Maker microworld to the non-computational seemed less problematic than the transition from two-spinners to two-dice. The situated abstraction, 'totals in the two-spinners gadget appear more often in the pie chart because they appear more often in the workings box', was available as a sense-making mechanism when they began work with the two-dice gadget. However, the meaning which drew on their everyday meanings for fairness of dice was initially more reliable. The webbing that allowed Anne and Rebecca to construct new meanings for the totals of two-spinners and then re-construct a similar meaning for the totals of two dice involved some specific features of the Chance-Maker microworld. As with the co-ordination of global meanings reported in Chapter Eight, the **repeat** primitive and the graphing tools were fundamentally important. Below, I list other aspects of the Chance-Maker microworld, which were featured in the co-ordination of meanings for the totals in the two-spinners and two-dice gadgets.

- Activity with previous gadgets had encouraged the abstraction of situated global meanings, essential in the construction of meanings for these compound events.
- The relative simplicity of the two-spinners gadget made the task of finding all the combinations easier than would have been the case had they been introduced immediately to the two-dice gadget.
- The dynamic action of the arms of the two spinners emphasised the difference between, say, 1+3 and 3+1.
- The format for the entries in the workings box, and the fact that they were different from those in previous gadgets, emphasised the need for a new approach to the analysis of this gadget.
- The default value of the workings in both gadgets encouraged engagement with the workings by intentionally omitting certain totals completely.
- Interaction with the workings drew attention to the fact that some totals were more frequently represented than others.
- Intervention where necessary encouraged seeing fairness in terms of equal representation of simple events in the workings box.

9.3. SUPPORTIVE EVIDENCE AND VARIATIONS FROM TWO OTHER CASE STUDIES

In this section, I draw on the interactions of two other pairs as they constructed new meanings for compound events. It is not necessary to present these cases in as much detail as I have done for Anne and Rebecca. The detailed analysis of these cases can be found at the world wide web address: http://www.warwick.ac.uk/wie/staff/DP.htm.

9.3.1. The Case of Donna and Rose

In her pre-interview, Donna had said that the totals of two dice, were equally easy or hard to obtain. Similarly Rose appealed to unpredictability to explain:

> "Well, any number could come out of each dice, cos they may not always be the same, however hard you shake them, so one might be a two and a three and one might be a five, but they might not always come out the same."

After some discussion in which I tried to help Rose to see totals as made up of several throws, Rose nevertheless stuck to her notion of unpredictability.

> "Not always, but I don't think it is really, cos you can get it like with any number really .... They're all about the same as each other."

There was some evidence later in the interview that Rose, perhaps because of the nature of my questioning, was beginning to break totals down into simple events.

> "Five maybe cos you can get three and two and you can get two, but if it's five you can only use two and three. Or four and one, there's two ways."

After 100 trials with the two-spinner gadget, Donna and Rose noticed that there was no 1 and no 5 in the results (3.10.3) (Figure 9.24).

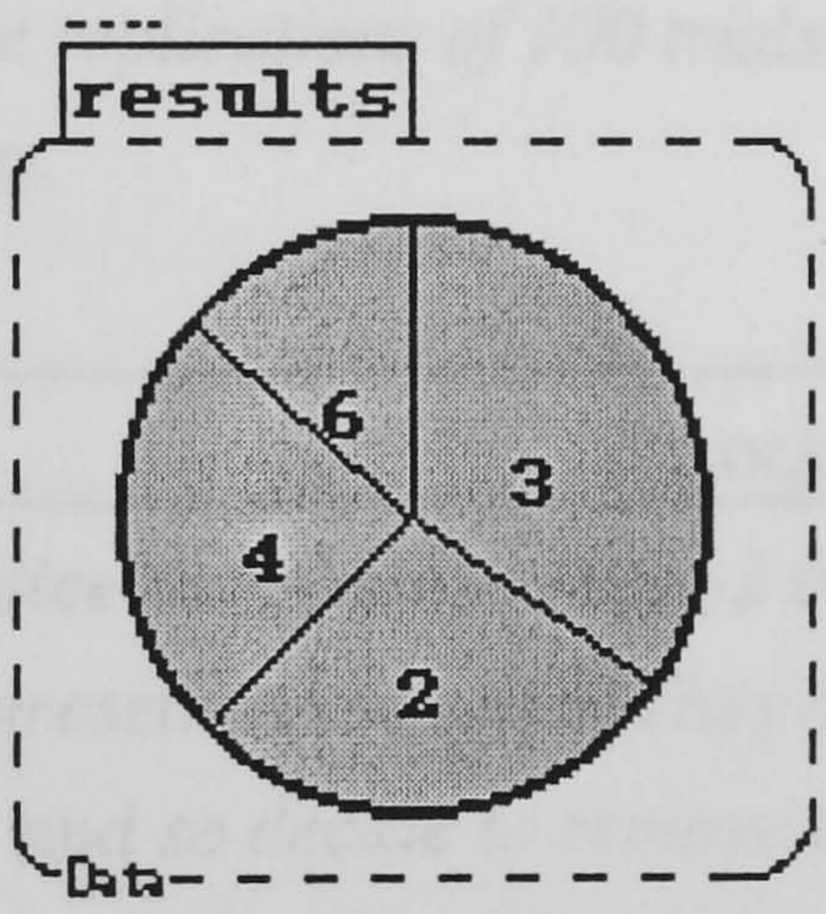

Fig. 9.16 : Donna and Rose noticed that there was no 1 and no 5

They quickly realised that 1 was impossible and calculated that it should be possible to obtain a 5 with a 2 and a 3. I asked whether any of the totals were easier to obtain than others on real spinners.

> Donna refers to the results so far and says, "It seems that there is. I think 2 is easiest to get." Rose explains why she thinks no total is easier, "I don't think it's easier to get any number, because they come out at random so they are not always going to come out as you think they are. So you might think that 2 is the easiest but it isn't always." Donna, realising that we are not talking about the computer's spinners, amends her view, "I don't think so *(i.e. that there is any total easier or harder)*, I think you've got a fair chance of each one."
>
> (3.10.5)

Donna argued that one could not tell whether the two-spinners gadget was working properly.

> "You don't know. Because next time we do it we could get a whole load of 5's. It's coming out at random, so you don't know."
> (3.10.6)

Both Donna and Rose believed that all totals for the two-spinners were equally likely. Below I trace the construction of meanings in the subsequent activity with the two-spinners gadget (they did not progress onto the two-dice gadget).

Para	Conjectures (C), Initial Meanings (IM) & Situated Abstractions (SA)	Critical Interventions
3.10.5 to 3.10.6	R: You don't know whether the two-spinners gadget is working properly as it is random (IM) D: The totals for two spinners are equally likely because the spinners are fair (IM)	
3.10.7 to 3.10.8	*Activity: After three replications of 100 trials, D & R are concerned that no 5's ever seem to appear.*	
3.10.8		Look carefully at the workings.
3.10.8	*Activity: D & R notice that there is no 3+2 in the workings. They observe that other totals are represented but not 5. They discuss how a total of 3 is in fact represented twice, and so decide to remove one of these. They edit the workings by replacing a 1+2 by a 2+3. After 100 trials, the 4 appeared most often.*	
3.10.8 to 3.10.9	4's might occur most because it has two ways in the workings and not 1 like the other totals (C) Repeating more trials might make the pie chart even (C)	
3.10.9	*Activity: D & R repeated 600 trials. 4's were still most common.*	
3.10.9	Perhaps strength is important (C)	

3.10.10	*Activity: Changing the strength to 60 made no difference when 600 trials were repeated once more. They returned to the previous workings,* ***choose-from [1+1 1+2 2+1 2+2 3+1 3+3]*** *and repeated 1200 trials. There were again no 5's in the results. They edited the workings to include 2+3 again instead of 2+1. They tried to change the strength once more.*

Table 9.2 : A trace of Donna's and Rose's construction of meanings for compound events

Because Donna and Rose were unable to construct a situated abstraction which allowed an alternative meaning of fairness, they never continued their work with the two-dice gadget. For Donna and Rose, the visual image of the symmetry of the spinners was so powerful that phenomena were interpreted in terms of that meaning. If the data did not conform to that meaning then they had to find an explanation for why the phenomenon was misbehaving.

In contrast, the previous two pairs had been able to re-construct a new meaning for fairness, defined in terms of the representation of all the possible simple events equally in the workings box. This was the meaning that eluded Donna and Rose. For them, the workings contained unfair representation of the total 3 since it appeared twice in the workings (1+2 and 2+1) whereas other totals, like 2, were only represented once. For some reason, they never noticed that the total 4 was also represented twice.

Donna and Rose were prepared to remove any distinction between 1+2 and 2+1 because the distinction introduced unfairness into the workings box. By missing the corresponding unfairness in the total of 4, they never attained an even pie chart, and, as has been observed many times, they searched for other explanations such as the strength of throw.

9.3.2. The Case of Neil and Gurdev

The third case, Neil and Gurdev, which I have chosen to illustrate support for, and variations on, the case of Anne and Rebecca, is interesting because of the boys' distinct initial meanings. When discussing the totals of two dice in his pre-interview, Neil continually referred to specific combinations, often doubles, rather than the totals. For example, when I asked him if it was possible to get all the totals between 2 and 12, Neil replied:

> "Yes, you can like 5 and 5, 2 and 2, 3 and 3, 4 and 4."

When I asked if there was one total which was harder to get than the others, Neil replied:

> ".... like 6 and 6, that kind of thing ....I just think the 3 is harder to get because most of the time, it always lands above 1 and above 3, like so *(and he tosses the two dice to justify his case)*."

Again when I asked him if any total is easier to get than the others, Neil replied:

> "2 2. I would say probably either 2 2 or 5 5 .... Because the 5, it's always going to come up, most of the time. It always comes up and, like the 2, whenever you toss, it always appears to come up."

Whenever I carefully used words to try to suggest a compound event, such as a total score, Neil interpreted my words in terms of specific combinations. So a total of 4 became a 2 and a 2, or a 3 and a 1 but both possibilities were never held in mind simultaneously.

In contrast, Gurdev suggested that a total of 2 and a 12 for two dice was harder to get than the other totals. He picked out 12 as harder than the others:

> "Yes, well I think it's 12 because you're aiming for two 6's."

Gurdev also thought a total of 2 was hard:

> "Cos even if you do get a one you might get a different number, like the 12 and two 6's .... Just that if you got a 1 and a 1, that's 2 and you could get a 6 and a 6, that's 12 Or you could roll the dice and it still might land on a 1 or a different number and then it might still land on the 6, but on a different number."

Gurdev saw all other totals as equally likely:

> "Cos, every time you spin it, it might go different." I ask Gurdev again, "So you'd say that the 2's and the 12's were harder but all the other totals were the same?" Gurdev replies, "Yes."

Gurdev appreciated that a total of 2 or 12 was hard because in each case there is only one way that it can be achieved. However, he did not continue this line of argument to differentiate between other totals. Instead he appealed to an unpredictability meaning.

In the early interactions with the two-spinners gadget, Gurdev indicated that he believed the totals for the two-spinners were equally difficult or easy to obtain (8.10.2). Perhaps surprisingly, he did not, at this stage, attempt any analysis of combinations. One interpretation is that the physical appearance of the spinner, with its equal sectors, was so strong that it dominated any other meanings.

In Table 9.4, I trace the construction of situated abstractions as the two boys work with the two-spinners and two-dice gadgets. Because, Gurdev's actions and articulations were so influential on Neil's construction of meanings, I occasionally enter Gurdev's statements into the critical interventions column.

Para	Conjectures (C) and Situated Abstractions (SA)	Critical Interventions
8.10.3	*Activity: They repeat 100 trials with the default setting and find no 5's.*	
8.10.3		G: There is no 3 plus 2 in the workings
8.10.4 to 8.10.5	*Activity: They change the workings to include all nine combinations. After 100 trials, the pictogram shows the classic triangular pattern with 4 as the most frequent outcome.*	
8.10.6	G&N: The picture shows most 4's because there are more ways of adding up to 4 (SA) G: Real spinners behave like these spinners (SA)	
8.11.1 to 8.11.5	*Activity: They repeat 100 trials with the two-dice gadget. The pictogram shows various totals are missing and 3 as the most frequent total. The workings are changed to include all 36 possible combinations. They repeat 100 trials. The pictogram shows 7 as the most frequent total.*	
8.11.5 to 8.11.6	N: There are six ways of getting 7 on the two-dice gadget. Smaller totals occur less often because the first dice might already be too big. (SA)	What do you think would happen if you did 1000 trials?
8.11.7	*Activity: They repeat 1000 trials. The pictogram shows 8's slightly ahead of 7's but otherwise the frequencies of the totals form the classic triangular shape.*	
8.11.7 to 8.11.8	N: The workings are always one step ahead. (SA)	How many ways of making each of the totals?
8.11.8	G: The workings are stepped like the triangular pictogram (SA) G: For real dice, the frequency of a total increases as the number of combinations that add up to that total increases (SA)	
8.11.9		G: There is 5 and 3 (as well as the double)

8.11.9	N: There are more ways of getting 8 than most other totals (even for non-computational dice) (SA)	

Table 9.3 : A trace of Neil's and Gurdev's construction of meanings for compound events

It must be admitted that Neil was influenced, at times heavily, by interventions from Gurdev and myself. Nevertheless, the exuberance with which he declared "I've got it now" and the expression of a reasonably coherent and original explanation, may suggest that he really had constructed a situated abstraction that 7's occurred more often than other totals and that the domain for the validity of this situated abstraction extended into the non-computational world. We can state with some confidence that Gurdev had made this abstraction and understood the generality of the domain of its application.

The new meanings constructed by Neil and Gurdev emerged in similar ways to the previous three cases. Gurdev was never concerned that 1+2, for example, might be the same as 2+1. These were self-evidently different outcomes on the two-spinners gadget, and on the two-dice gadget. As a result, Gurdev, and then Neil, were able to co-ordinate meanings of fairness, which resulted relatively quickly in an acknowledgement that totals on the two-spinners and two-dice gadgets were not equally likely.

The triangular pictogram image was especially powerful for Neil and Gurdev. Though this was not so much the case for the other two cases reported, some of the children in the remaining five cases were also supported by the triangular image. Neil and Gurdev used the triangular pattern as a way of describing the pattern in the workings box, and so put together a meaning which connected the pictogram, the pie chart, and the workings.

9.4. COMPARISON OF THE THREE CASE STUDIES

Three case studies have been presented. As a whole, they demonstrate the webbing between local and global meanings for behaviour, and the external resources in the Chance-Maker microworld, guided by my own interventions, and resulting in the co-ordination of new meanings, associated with compound events. These new meanings take the form of situated abstractions.

9.4.1. Situated Abstractions in the Three Case Studies

In this section, I look briefly at some of the situated abstractions across the three case studies.

Different totals on a gadget occur different numbers of times

The most elementary of observations, abstracted directly from activity within the Chance-Maker microworld, was that the totals for two-dice or two-spinners gadgets did not appear equally often. The role of the pie chart and pictogram tools was fundamental in making this observation.

The observation would generate different responses. We saw how both the Anne / Rebecca and Donna / Rose pairs responded initially by increasing the number of trials and by amending the workings to make the pie chart even. These strategies were presumably based on the situated abstractions for long term behaviour, drawn from their work with previous gadgets (i.e. the number of trials controls the evenness of the pie chart; the workings control the evenness of the pie chart). Donna and Rose never really managed to move beyond that stage. They became stuck into a procedure of trying to make sense of the uneven pie charts by changing the strength or amending the workings.

The initial meaning that the totals in the two-spinners and two-dice gadgets should appear equally often was well established. Anne and Rebecca were able to consider other meanings for fairness through my intervention that perhaps it was unfair to include some outcomes of the two-spinners gadget more than once. The Neil / Gurdev pair seemed to have few difficulties over the notion of fairness, though in Neil's case, there is a strong sense that he was helped enormously by his partner, Gurdev.

The frequency of representations of a total controls the size of its sector in the pie chart

This situated abstraction is a refinement of 'the workings controls the pie chart' situated abstraction. Nevertheless, the refinement is not trivial. It was necessary to process the workings so that individual outcomes of trials could be re-aggregated into compound events. For example, a 4 on the two-spinners gadget had to be seen as the accumulation of 1+3, 3+1 and 2+2. Most pairs achieved this step through engagement with the workings box of the two-spinners gadget. It was then usually less difficult to make a similar step for the two-dice gadget.

In most cases, the move towards interaction with the workings was made when it was noted that some totals were absent in the pictogram or pie chart (more often the former). The decision to exclude some totals from the default setting of the workings was fully justified.

Through amending the workings box, the children had a vague impression that some totals were more frequently represented than others. This impression was called upon when they tried to explain the appearance of the graphs, which often contradicted their expectation that the totals would be equally likely.

Along the way, the children needed to decide whether to include both versions of an outcome pair. Was it necessary to include both 1+2 and 2+1, for example? For the boys, this step was relatively straightforward. Anne and Rebecca needed some help to see that these outcomes corresponded to two distinct outcomes on the two spinners. Donna and Rebecca never satisfactorily made this distinction, since they were prepared to drop the 'extra' outcome when the results did not meet with their expectations. Nevertheless, most of the eight pairs in Iteration 3 were supported by the close connection between the dynamic appearance of the spinners and the representation of outcomes in the workings box.

The workings are stepped like the triangular pattern in the pictogram

Neil and Gurdev, and some of the cases not reported here, found the triangular pattern on the pictogram most impressive. The stepped image became a resource which Neil and Gurdev used to describe and make sense of the workings. Other pairs used this image to link the two-spinners gadget and the two-dice gadget, both generating triangular images in their pictograms.

9.5. DISCUSSION

New meanings for compound events, in the form of situated abstractions, emerged out of the local and global meanings associated with simple events. The children in all eight case studies articulated their global meanings by using, throughout their activity with the two-spinners and two-dice gadgets, large numbers of at least 100 trials and often as many as 1000. They also drew constantly on situated abstractions such as 'the workings box controls the evenness of the pie chart'.

Most of the children came to this activity with strong initial meanings, based on symmetry and experience of short-term behaviour, that the totals should be uniformly distributed. Even after amending the workings to include all possible outcomes, the charts were not as expected. The issue then was whether the juxtaposition of their initial meanings for fairness and the situated abstractions from their earlier experience would co-ordinate to create a new situated abstraction for the behaviour of compound events. In the case of one pair, Donna and Rose, this did not seem to be the case. Their intuition that the outcomes should be equally distributed held firm even at the end of the session. For the other pairs, there was evidence of co-ordination, which resulted in new meanings for compound events.

The notion of fairness was adapted to yield a situated abstraction in which each outcome was represented once in the workings box; the totals were then discerned by accumulation and counting within the workings box. This procedure forged a new connection between the workings box and the chart, often the pictogram, in which the accumulation of outcomes which represented a particular total controlled the appearance of the pictogram.

In Figure 9.17, the webbing process, which enables this co-ordination is sketched out.

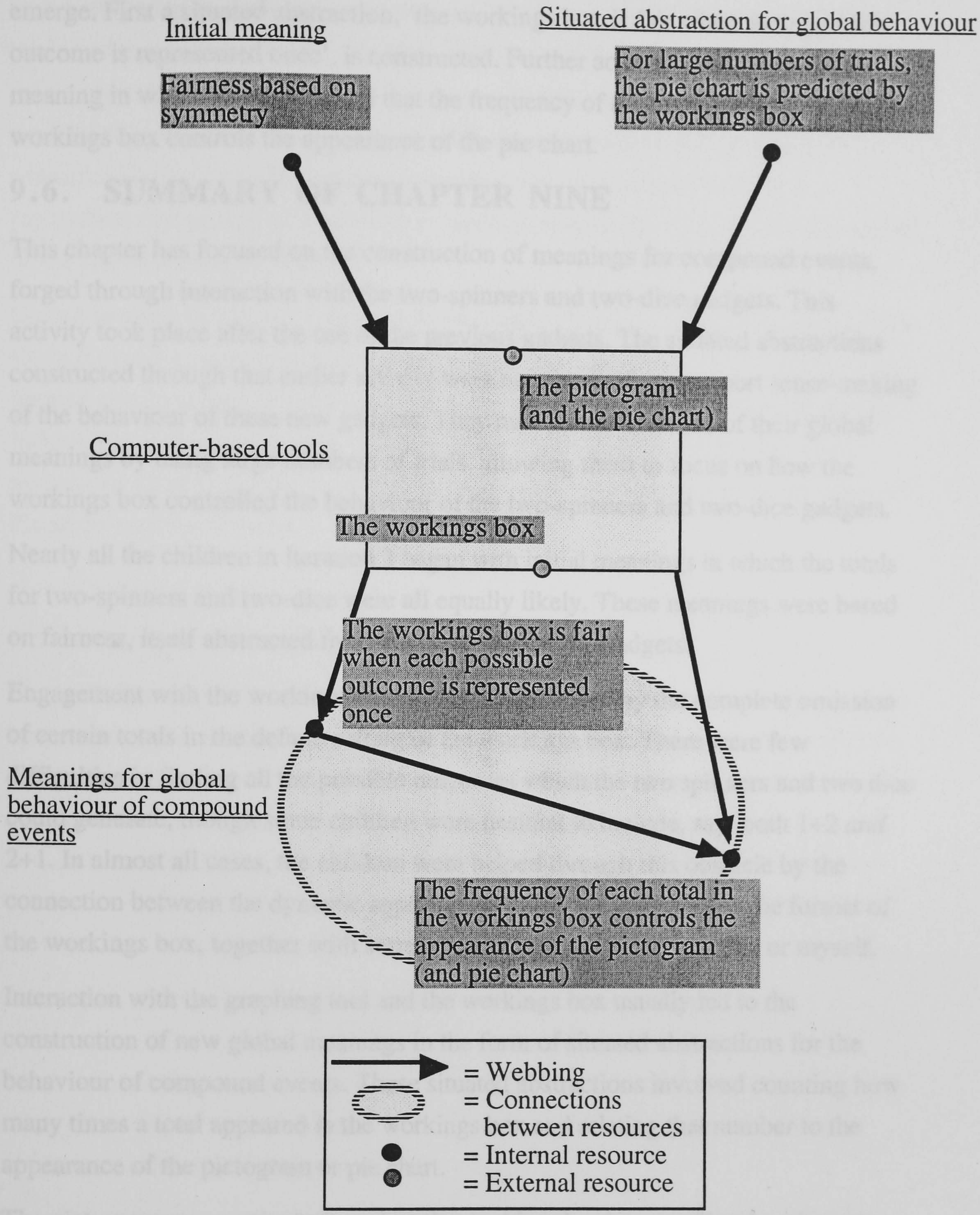

Fig. 9.17 : The construction of global meanings for the behaviour of compound events

In Figure 9.17, activity with the two-spinners or two-dice gadgets cues the local meaning of fairness and the situated abstraction constructed from previous activity for the predictability of the pie chart in the long term. These local and global meanings are illustrated at the top of the diagram, and they are shown to be connected, because of the sense-making activity, with the computer-based tools. Out of this activity, new global meanings for the behaviour of compound events emerge. First a situated abstraction, 'the workings box is fair when each possible outcome is represented once', is constructed. Further activity results in a new meaning in which it is recognised that the frequency of the various totals in the workings box controls the appearance of the pie chart.

9.6. SUMMARY OF CHAPTER NINE

This chapter has focused on the construction of meanings for compound events, forged through interaction with the two-spinners and two-dice gadgets. This activity took place after the use of the previous gadgets. The situated abstractions constructed through that earlier activity were now available to support sense-making of the behaviour of these new gadgets. They made immediate use of their global meanings by using large numbers of trials, allowing them to focus on how the workings box controlled the behaviour of the two-spinners and two-dice gadgets.

Nearly all the children in Iteration 3 began with initial meanings in which the totals for two-spinners and two-dice were all equally likely. These meanings were based on fairness, itself abstracted from the symmetry of the gadgets.

Engagement with the workings box was often provoked by the complete omission of certain totals in the default setting of the workings box. There were few difficulties in finding all the possible outcomes which the two spinners and two dice could generate, though some children were hesitant to include, say, both 1+2 *and* 2+1. In almost all cases, the children were helped through this obstacle by the connection between the dynamic appearance of the two spinners and the format of the workings box, together with some support from working partners or myself.

Interaction with the graphing tool and the workings box usually led to the construction of new global meanings in the form of situated abstractions for the behaviour of compound events. These situated abstractions involved counting how many times a total appeared in the workings box and relating that number to the appearance of the pictogram or pie chart.

The pictogram was particularly supportive in this role, since it offered a triangular pattern which was often connected to the stepping in the frequencies of the totals in the workings box. It is interesting to note that, in contrast, the proportionate nature of the pie chart had been more helpful in the previous work, when the pattern in the sizes of the sectors had been unconnected with the ordering of the various outcomes.

The new situated abstractions for the behaviour of compound events seemed to be variations on those generated for simple events. The crucial difference was that the totals had to be accumulated mentally within the workings box. Whereas listing all the combinations that could occur for either of the two gadgets was relatively straightforward for nearly all children (with the exception of how to deal with examples like 1+2 and 2+1), the new meanings were constructed out of a recognition that groups of these simple events needed to be collected to make sense of the appearance of the charts.

Once a situated abstraction had been formed which 'explained' the behaviour of the totals for the two-spinners or two-dice gadgets, most children (though not all) seemed able to extend the domain of its applicability to non-computational settings. Indeed, the transition for some children from the two-spinners to the two-dice gadgets was less easily accomplished as evidenced by the needed to repeat some of the sense-making activity.

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27 In Figure 9.14, the pictogram is too small to accommodate all the throws. Hence, several rows appear to 'fill' the chart. Usually, children would re-scale the diagram using the 'worth' box so that all the throws would fit. In this instance, the focus is on the empty rows and so re-scaling was not necessary.

CHAPTER TEN

Conclusions and Theoretical Considerations

10.1. OVERVIEW

Chapter Ten brings the thesis to a conclusion. I have used the term, 'window', as a metaphor to describe the way in which the computer screen offers insights into the children's meaning-making as they use the tools of the Chance-Maker microworld. As I gaze through that window, I can attend to the findings of this study at a variety of grain sizes. In the middle-distance, I look upon the broad aims of the thesis. In the immediate foreground, I inspect the fine detail. In the far distance, I gaze out upon the landscape to discuss how these findings might contribute to the development of a theoretical framework.

The window metaphor provides a structure for the final chapter in which I summarise the findings corresponding to the middle-distance and the foreground of view before discussing findings on the landscape in the form of a theoretical framework, which may tentatively incorporate meaning-making beyond the confines of the Chance-Maker microworld, possibly spanning other mathematical domains.

10.2. SUMMARY OF FINDINGS RELATED TO THE BROAD THEMES OF THIS STUDY

I begin the discussion by looking through the window of the computer screen into the middle distance, where I can focus on how my analysis of the children's activity with the Chance-Maker microworld enables me to elaborate on the broad aims of the study.

A top-level intention of this study was to explore the feasibility of building a domain in which formal expressions of mathematical concepts are introduced into informal domains of learning. This aim is discussed in section 10.2.1. A second broad aim was to identify the nature of the intuitions that children use to construct meanings for random behaviour, and how these are shaped by the structuring resources in the environment. This aim is discussed in section 10.2.2. In particular, I wished to examine the extent to which the newly constructed meanings are contingent upon these resources. This aim is discussed in section 10.2.3.

10.2.1. Connecting the Formal and the Informal

I wish first to review an overarching principle which has informed the direction, and therefore the findings, of this study. This study has located the construction of meanings for the stochastic in a grey area lying at the intersection of several domains of knowledge. Whilst the stochastic has been formalised as a mathematical domain in recent history, randomness is also experienced in the everyday, part of the phenomenological scene that we encounter in natural settings, akin to our encounters with scientific phenomena. Yet, unlike many aspects of science, we are rarely in a position where we can experiment with random phenomena. More likely, we will merely *respond* in various ways to the vicissitudes of randomness, without gaining any sense of control. Randomness is experienced as that which is unsteerable.

The stochastic is therefore located across both formal mathematics *and* informal settings – yet these two arenas are rarely brought into juxtaposition. This study has described the story of how a domain has been built in which formal and informal views of the stochastic have been brought together. Perhaps the workings box is the richest and most elegant encapsulation of how the distinction between the formal and the informal can be blurred. The workings box represents simultaneously both a formal probability distribution *and* a control point of informal activity.

Understanding the stochastic also two dimensions: the intuitive and the operational. I have presented evidence that children appear to have partitioned random phenomena into something quite separate from other everyday phenomena. The stochastic is discriminated as that which does *not* have the characteristics of the deterministic. By bringing the formal and informal into closer proximity, I have created a pedagogic approach which appears to have the potential to empower children to connect their intuitions for deterministic and stochastic phenomena, activating sense-making apparatus, which constructs new meanings for the stochastic.

The stochastic has proved to be a very rich domain for the study of issues which pervade the fringes of science and mathematics. I have been able to operate in a territory infrequently visited by other researchers. I have been able to report on *young* children's intuitions for the stochastic and, in particular, on their meanings for randomness as abstracted from everyday activity. I have been able to discuss in some detail how those meanings are shaped by the specific features of the microworld, and draw some conclusions about the design of such software. How do these findings relate to the work of previous researchers?

10.2.2. The Construction of Meanings for the Stochastic in the Chance-Maker Microworld

Piaget and Inhelder argued that children come to recognise random mixtures as those which can not be explained by operational thinking; in particular randomness can not be reversed, a defining characteristic of operations. There is support for this notion in this study. It seems that the children characterised their meanings for the stochastic through the absence of properties, apparent in deterministic phenomena. If a phenomenon was *not* predictable, *not* controllable, *not* regular in its behaviour, then it was random. The exception was fairness, which for some children was deterministically based in the sense that fairness was sometimes inherent in the symmetry of the device and this symmetry would *cause* fair results. Presumably all these notions were abstracted from the everyday use of terms like fairness and randomness in experiencing short-term unpredictable and uncontrollable behaviour.

Piaget and Inhelder argued that the construction of probability is dependent upon formal operations, because of the link with concepts of proportion and combinatorics. In this study, we have seen how children were supported in the construction of situated abstractions of proportion by the pie chart tool. Meanings for the combinations of outcomes were simultaneously supported through connections between the workings box and the appearance and use of the gadgets themselves, even in the relatively complex case of totals of two spinners and two dice. The point here is not to refute Piaget and Inhelder but to stress that the technological tools available gave a new kind of access to the concepts of proportion and combination. There is a sense in which the Chance-Maker microworld moves the goal posts: the goal changes from the construction of formal operations to the meaningful use of tools.

10.2.3. From Misconceptions to Situated Abstractions

The analysis in this study suggests that the children's local meanings are used initially to make sense of long-term behaviour. The local meanings then appear to be misconceived. Yet, meanings such as unpredictability, unsteerability and so on, are the only resources available to the children as they attempt to make sense of the behaviour of the gadgets. One research approach would be to add these misconceptions to the ever growing list of ways in which people, in this case children, behave irrationally when making judgements in the stochastic domain.

Such a conclusion would be consistent with much of the work by Kahneman and Tversky in which they catalogue heuristics such as representativeness and availability and discuss the inherent bias in the use of these heuristics. Such an approach would also be in keeping with the work of Konold, who has reported a tendency for people to attend to the actual outcomes, rather than the probabilities in a situation.

This study has shown though that such an approach over-simplifies the intuitional resources available to children, missing the point that these meanings may have the characteristics of misconceptions when the circumstances lie beyond the child's area of competence, but, set in a carefully designed domain of abstraction, they become the raw materials of new meanings. The notion of misconception ignores the potential for those same intuitions to act as a springboard for successful sense-making. My argument then is that misconceptions are manifestations of a research methodology in which people are asked to perform at levels outside of their area of competence and without tools which would help them to make sense of that domain.

This study has identified precisely the sorts of tools, which enable young children to make sense of the long-term behaviour of random phenomena, discriminating a new domain of applicability for local meanings and constructing new global meanings with new domains of applicability. The children in this study were able to use tools which helped them to make familiar aspects of randomness and probability. The construction of situated abstractions for concepts like proportion, the combination of outcomes that make up a specific total and the Law of Large Numbers illustrate the notion of concretion proposed by Wilensky. I am not suggesting that the children *understood* randomness and probability, rather that they constructed new meanings, which were situated within the Chance-Maker microworld, but whose domain of applicability was extended, possibly to the non-computational, certainly to an increasingly wide range of gadgets.

Nunes observed children and adults applying knowledge, constructed from meaningful but specific contexts, to other new contexts, suggesting that the situated knowledge possessed a potential for generalisation. The re-use of situated abstractions in this study was rarely smooth and trivial. The features of the new context did not necessarily cue those situated abstractions until various other meanings, some of which were long established, had been found to lack explanatory power, given the behaviour of the gadgets. The evidence in this study then suggests that situated abstractions offer internal resources, which can be, but may not be, applied to new contexts.

10.3. SUMMARY OF FINDINGS RELATED TO THE SPECIFIC AIMS OF THIS STUDY

I now switch my attention to look through the window of the computer screen at the immediate foreground. Each of the three broad findings above can be elaborated further by summarising the findings as they relate to the specific aims of this study. In Chapter Three, I framed the aims as answers to two sets of questions. Set A referred explicitly to the meanings that children construct *for* a domain of stochastic abstraction, whereas set B referred to the meanings that can be constructed *in* a domain of stochastic abstraction. I propose to summarise the detailed findings from this study by consideration of each of these questions in turn, re-stated at the head of each sub-section.

10.3.1. Meanings For a Domain of Stochastic Abstraction.

A1. *In designing a Boxer-based domain which emphasises meaningfulness, rather than rigour, generality or economy, what do formalisms of the stochastic phenomena, such as randomness and distribution, look like?*

Formalisms which express the generality and rigour of mathematics must divorce the formalism from a specific setting. Yet it is the features of the setting that promote the sense of purpose and utility which are essential in the initial stages of sense-making activity. On the other hand, if we present context-based formalisms, they will inevitably lack the rigour of the underlying mathematics, and the presence of surface features of the setting will fail to suggest its generality . How can children construct a sense of the rigour and generality of the mathematical concepts if they are not introduced to those notions. This is the fundamental dilemma in the design of a domain for abstraction.

A resolution of this dilemma was to present (the italics give specific examples from the Chance-Maker microworld):

- a range of formalisms, each of which represented situated versions of the 'same' central mathematical concept *(the workings box in each gadget)*,
- a range of formalisms, across which there exists a commonality (i) between the language used to express the mathematical concepts, and (ii) between the ways in which the gadgets are activated and manipulated *((i) the **choose-from** primitive (ii) click-and-play)*,

- formalisms, which could act as a control over the mathematical components of the informal activity *(the workings box)*,
- executable formalisms which could be amended by the child, promoting ownership and the facility to explore and validate (or invalidate) personal conjectures *(the workings box)*,
- controls over the mathematics alongside 'redundant' controls of non-mathematical aspects of the informal activity, enabling the discrimination of the different types of control *(the strength control)*.

The above characteristics of the solution manifest themselves in the Chance-Maker microworld in the form of workings boxes within gadgets, which:

- offer situated versions of the notion of probability distribution, versions which possess the surface features of the gadget *(for example, the workings box of the coin gadget refers directly to the faces of the coin)*,
- make explicit mention of the **choose-from** primitive, an encapsulation of the concept of randomness and a common link across the various gadgets,
- can be amended by changing the possibility space to explore the effects of such changes and so test out their own conjectures for the gadget's behaviour,
- offer stochastic control of the outcomes for the gadget,
- can be discriminated as control mechanisms from the sort of control offered by the strength control, which has no bearing on the outcome but does influence some aspects of the dynamic appearance of the gadget *(for example, how long the dice rolls or the spinner spins)*.

A key principle then underlying the design of the Chance-Maker microworld is that the formalisms should be designed to offer direct control over behaviour related to the central mathematical concepts embedded within the domain. The workings box acts, at one and the same time, as an encapsulation of probability distribution and as a control mechanism over informal exploration.

A2. When children use Boxer-based versions of artefacts like dice and coins, what structures should we embed in the domain for stochastic abstraction in order to facilitate the articulation of intuitions about how those artefacts behave?

There were two design heuristics which focused on encouraging engagement with the Chance-Maker microworld, namely:

(i) designing for purpose, and

(ii) designing for familiarity.

I will summarise the findings, which relate to aim A2, by focusing on each of these heuristics in turn.

(i) The workings box formed one part of the overall design of the gadgets. Its use as a controlling mechanism for informal activity owed part of its success to the transparent connection between the workings box and the stated purpose of the activity. The task for the children was to identify which gadgets were working properly and to mend those which did not. The design of the Chance-Maker microworld revolved around that explicit purpose. The task led them eventually to the editing of the workings box but this activity was preceded in all cases by initial sense-making, which centred on how the gadget behaved compared to their expectations.

(ii) Expectations of behaviour were cued by the way that the gadgets were designed to appear familiar, tapping into children's cultural experiences with devices like coins, spinner and dice, encouraged by their familiar surface features. The gadgets looked, and behaved on activation, just as the children would have expected.

The effect of cultural and surface familiarity was to cue intuitions about the behaviour of such phenomena abstracted from everyday settings. These intuitions were articulated through the children's discussions between each other, and in response to my questions, and through the activity with the gadgets, in the form of button-clicks and amendments to the workings box. Through observation of these actions, it was possible to identify manifestations of local meanings for the behaviour of the gadgets, meanings such as unpredictability, unsteerability, irregularity, fairness and computer-in-control.

A3. Through the iterative design, which structures will facilitate the forging of new connections between intuitions (e.g. of randomness) and formalisms (such as representations of distribution)?

New meanings were articulated through use of the workings box in conjunction with three tools and resources.

(i) The **repeat** primitive

The **repeat** primitive facilitated the use of large numbers of trials.

(ii) The results box

The results box allowed observation of children's irregularity meaning, and later children were able to check that the behaviour of a gadget was still random from result to result even though it was predictable when viewed in aggregation.

(iii) The graphing tools

The graphs and charts played a fundamental part in enabling the children to 'see' the patterns in aggregated results. The pictogram was an important structuring resource in the construction of meanings for the totals in the two-spinners and two-dice gadgets. Children made sense of the order in the likelihoods of successive totals from 2 upwards by reference to the ordered triangular pattern in the frequencies of the outcomes for the two-spinners and two-dice gadgets.

The pie chart was fundamentally important in supporting children's construction of global meanings for the behaviour of the coin, spinner and dice gadgets. The proportionate nature of the pie chart enabled children to focus on the increasing evenness of the pie chart as the number of trials increased (for a uniform workings box).

10.3.2. Meanings In a Domain of Stochastic Abstraction

B1. *When young children interact with Boxer-based representations of everyday stochastic phenomena (dice, coins, spinners .....), what expressions of their informal intuitions of stochastic behaviour do we observe?*

Children articulated five local meanings for the behaviour of Chance-Maker's gadgets:

(i) unpredictability,

(ii) unsteerability,

(iii) irregularity in results,

(iv) fairness, and

(v) computer-in-control.

In the case of the two-spinners and two-dice gadgets, two meanings for fairness were expressed:

(i) The total of two spinners should be fair (and so equiprobable) because the spinners are symmetrical, unpredictable and unsteerable,

(ii) The totals should be fair if we can include all the possible outcomes in the workings box.

Meanings emerged out of sense-making activity which searched first for deterministic causes for the behaviour observed. Conjectures that the children tested included (the italics refer to an illustrative evidence from the case accounts):

- the order of the outcomes in the workings box causes some numbers to appear more frequently *(Ray & Luke, 2.7.10)*;
- the result depends on the starting point of the gadget *(Ray & Luke, 2.3.2)*;
- the result is determined by the strength used *(Anne & Rebecca, 6.3.2)*;

The behaviour of the gadgets was described as random when either:

(i) evidence from the children's interactions with the gadgets was seen as inconsistent with deterministic explanations for that behaviour *(Steve & Richard, 4.3.2)*, or

(ii) the causal factors were recognised as lying outside of the child's control and so the behaviour was random as far as the child was concerned *(Anne & Rebecca, 6.7.7)*.

The five local meanings for behaviour were connected, but only weakly, as evidenced by:

- the apparent interchangeability of one meaning with another *(Anne & Rebecca, 6.7.2)*,
- how a slight variation in the surface features of a situation cued a different meaning *(Donna & Rose, 3.3.2)*,
- potential contradictions between the meanings, such as an unpredictable but unfair spinner, were not recognised as problematic *(Lynn, 7.3.1-2)*

The five local meanings appeared to be set apart from other components of a child's sense-making apparatus; in particular, the local meanings for random behaviour appear disconnected from meanings for deterministic behaviour. When children used these local meanings:

- they appeared to be self-explanatory,

- each meaning was used in support of another but was not associated with other pieces of knowledge,
- there was little or no reference to knowledge outside of these meanings — there appeared to be no further inferences that could be made — no deductions which might have implications for other aspects of the microworld.

B2. How do the structures within the domain of stochastic abstraction, built into Boxer, support the forging of situated abstractions?

Initially, local meanings were used by children in their attempts to make sense of both short-term and long-term behaviour. I have illustrated the co-ordination of existing local meanings for behaviour whereby new global meanings for behaviour are constructed through the forging of connections with computer-based resources. These global meanings became associated with long-term aggregated behaviour of the gadgets.

The pie chart tool played a fundamental part for most children; the proportionate nature of the pie chart supported the construction of situated abstractions for long-term behaviour. In contrast, the pictogram tended to place emphasis on the differences between the lengths of rows in the graph. Indeed, as the number of trials increases, the absolute difference between frequencies is likely to increase, reflected in larger variations between the lengths of the rows of the pictogram. One might suppose then that the pictogram image was consistent with the children's local meanings of unpredictability, whereas, in the pie chart, the sectors became closer and closer in size (when the workings box represented a uniform distribution), a surprising result which demanded explanation. The importance of the pie chart is reflected in the way that the situated abstractions below are expressed.

Global meanings for long-term aggregated behaviour of Chance-Maker's gadgets emerged out of the co-ordination of local meanings for stochastic behaviour and intuitive knowledge of cause and effect relationships (what diSessa calls 'an intuitive sense of mechanism'). The new global meanings took the form of situated abstractions, articulated as below. (The square brackets represent aspects of the situation which were not necessarily articulated, especially early in the sense-making activity, but which emerged as an important condition on the meaning. The italics reference an illustrative example from the case accounts.)

- the number of trials controls the evenness of the pie chart [the workings box is uniform] *(Anne & Rebecca, 6.8.9)*;
- the pie chart looks like the spinner [the number of trials is large] *(Ray & Luke, 6.7.8-9)*,
- less trials cause more variety amongst the sectors of the pie chart [the workings box is uniform] *(Anne & Rebecca, 6.9.6-8)*;
- the workings box controls the evenness of the pie chart [the number of trials is large] *(Anne & Rebecca, 6.9.3-4)*;
- the number of trials AND the workings box control the evenness of the pie chart *(Anne & Rebecca, 6.9.6-8)*.

There is an order to the above situated abstractions, given the way that the Chance-Maker microworld was used by the children. The specific features of the activity which determined this order are set out below.

The number of trials was first constructed as a controlling influence because the activity supported a move towards an increasing number of trials:

(i) in search of more information *(Ray & Luke, 2.7.2-4)*,

(ii) by accident *(Donna & Rose, 3.8.2-3)*, or

(iii) through my intervention *(Donna & Rose, 3.9.9)*.

The controlling influence of the number of trials was also discriminated through the need to explain the behaviour of the gadgets for a *large small* number of trials, where the children expected evenness in the pie chart but where in fact there was quite a high chance of unevenness.

Thus, the children abstracted a notion that the number of trials controls the appearance of the pie chart prior to a notion relating to the controlling influence of the workings box.

The workings box was constructed as a controlling influence because:

(i) there was an explicit visual connection between the appearance of the spinner and the workings box *(Ray & Luke, 2.7.2-4)*,

(ii) the children were drawn to amending the workings box by its default unfair setting *(Ray & Luke, 2.8.2)*, and

(iii) the number of trials failed to explain fully the unevenness of the pie chart

> when the workings box was non-uniform (when the workings box was non-uniform and the number of trials large, the pie chart was not even) *(Anne & Rebecca, 6.9.2).*

Finally, neither the number of trials nor the workings box were seen as offering complete explanations for the behaviour of the gadgets; a co-ordination of these two meanings resulted in a situated abstraction in which both the number of trials AND the workings box were seen as controlling influences.

New global meanings for the long-term behaviour of compound events were also evident. The main situated abstraction was articulated as 'the frequency of representations of a total in the workings box controls the size of its sector in the pie chart'.

This situated abstraction emerged out of activity in which the children aimed:

(i) to make the pie chart fair, and then

(ii) to make the workings box fair.

Initial meanings for the fairness of the spinners and the dice were so strong that the children usually amended the workings to include equal representations of each total. Using a large number of trials, an even pie chart was generated.

Except in the case of Donna and Rose, making the workings box fair was prompted and supported by one or more of the following factors:

- a recognition that each combination of the two spinners or two dice should be represented in the workings box *(Anne & Rebecca, 6.11.3-6)* ,
- a recognition that outcomes like 1+2 and 2+1 were different and should both be included in the workings box since each represented different throws of the spinners (or dice) *(Anne & Rebecca, 6.1.7-8),*
- the triangular appearance of the pictogram, which encouraged a linking between the frequency of representation of different totals in the workings box and frequency of occurrence of those totals in the results *(Neil & Gurdev, 8.11.7-8).*

B3. What are the features of the webbing process (including the children's intuitions as articulated within the Boxer microworld, the tools provided and the influence of other agencies such as other children and my own interventions), which determine the extent to which these situated abstractions become tools for the forging of new connections in related activity within other parts of the Boxer-based domain?

Whereas the formal perspective on mathematics presents an image in which mathematics is decontextualised, the informal view connects mathematics closely to the surface features within a context of activity. Chapter Two discussed the work of Lave and Nunes to illustrate the ideas of the situated cognition movement, and, in particular, how a strong interpretation of their theories proposes that knowledge is deeply, perhaps irrevocably, constituted within a setting, shaped by the available tools and resources. According to this view, randomness as constituted through activity with dice would be quite different from randomness as constituted through activity with spinners. The formal view of mathematics in contrast presents the construct of a random variable as de-contextualised.

This study has focused on this dichotomy by:

- reporting on children's sense-making activity as they moved from one context to another, attempting to construct meanings for the behaviour of successive gadgets in the Chance-Maker microworld,
- enabling the observation of how the specific attributes of one gadget shaped, or did not shape, the construction of meaning, not only for the behaviour of that gadget, but also for subsequent gadgets.

I interpret these observations as suggesting that:

- there was evidence that the domain of applicability for situated abstractions forged through activity with a gadget might be extended:

  (i) children re-used situated abstractions from a previous gadget whilst trying to make sense of the behaviour of a new gadget — for example, 'the higher the number of trials, the more even the spinner's pie chart' was usually extended to the dice *(Rose, 3.9.3-4)*;

  (ii) children applied situated abstractions to thought experiments pertaining to imagined game-like situations away from the computer — for example, almost all the children were able to re-use the situated abstraction that 7 was the most common total for the two-dice gadget to discuss imagined situations with dice at home *(Neil & Gurdev, 8.11.9)*;

- the situated abstractions, constructed from one gadget, existed as internal resources (alongside other local meanings, perhaps abstracted directly from prior experience), available, but not necessarily cued, during sense-making activity with subsequent gadgets *(Anne & Rebecca constructed the situated abstraction for the controlling influence of the number of trials on the coin's pie chart, 6.7.12, but then engaged in extended activity with the spinner before re-constructing a similar situated abstraction, 6.8.9)*
- sense-making activity with a subsequent gadget would usually focus initially on long-established meanings, but, as these meanings proved unreliable in the face of evidence from the Chance-Maker microworld, situated abstractions constructed from prior gadgets were cued as resources *(Anne & Rebecca, 6.7.12-6.8.9)*;
- once a situated abstraction had been found to be consistent as a sense-making resource across more than one gadget, that situated abstraction tended to be more easily cued in activity with other gadgets *(Anne & Rebecca, 6.9.6-8)*.

Before I discuss how these findings inform the development of a theoretical framework for meaning-making, I wish to set out the main limitations of this study, factors which constrain the elaboration of its aims.

10.4. LIMITATIONS OF THIS STUDY

10.4.1. Time

The amount of time spent with an individual child or with a pair of children was extremely limited. Typically each child was interviewed for about 30 minutes and each clinical interview lasted for about 2.5 hours. Longer periods of time may have allowed other meanings to emerge whilst it is possible that some meanings observed would have proved to be less significant than the interpretation given in this account.

10.4.2. Paired Children

For the clinical interviews, children worked in pairs. The advantages of this approach were discussed in Chapter Four. Nevertheless, there are clearly difficulties in separating individual constructions in this methodology. It is feasible therefore that meanings interpreted as co-constructed by the pair of children may in fact have sometimes been more significant for one child than the other. Though often such asymmetry was transparent through my probing questions, it is likely that some imbalances were missed.

10.4.3. Sensitivity of Findings to the Specific Features in the Microworld

Experience of designing microworlds in this study and in other contexts emphasises the sensitive nature of the relationship between the children's activity (and consequent construction of meaning) and the tools and structures made available within the microworld. The iterative design methodology helps to gain a feel for the nature of this relationship but it is necessary to interpret the findings with some caution.

10.5. TOWARDS A THEORETICAL FRAMEWORK

With these limitations in mind, I can now switch my attention one final time to gaze through the window of the computer screen, beyond the specifics of the Chance-Maker microworld, upon the landscape, where I can seek to develop a theoretical framework for the construction of meanings for the stochastic.

10.5.1. The Co-ordination of Meanings for Randomness

The situated abstractions for the long-term behaviour of the gadgets take the form of cause and effect relationships, where one specific factor, such as the number of trials or the workings box, causes a corresponding effect, such as the evenness of the pie chart. This is in stark contrast to the local meanings which describe attributes, linked to each other by association or inference. For example, the dice is unsteerable, so it is random, so it is unpredictable.

The cause-and-effect situated abstractions are not, however, disconnected from the local meanings. Local meanings become reliable as predictors of short-term behaviour but of reduced reliability for making sense of long-term aggregated behaviour (*reliability* in the sense used by diSessa). New global meanings take on increasingly high reliability as they are found to explain long-term behaviour better than longer established meanings and to be successful predictors of the behaviour of more and more gadgets (an example of what diSessa calls 'tuning towards expertise'). In this sense, the system of local meanings is re-structured whilst new global meanings are created afresh. The co-ordination of local meanings can be modelled as a condition, attached to, for example, the unsteerability meaning. If indeed the number of trials is large, the condition activates the controllability global meaning. A first attempt to schematise this model is depicted in Figure 10.1.

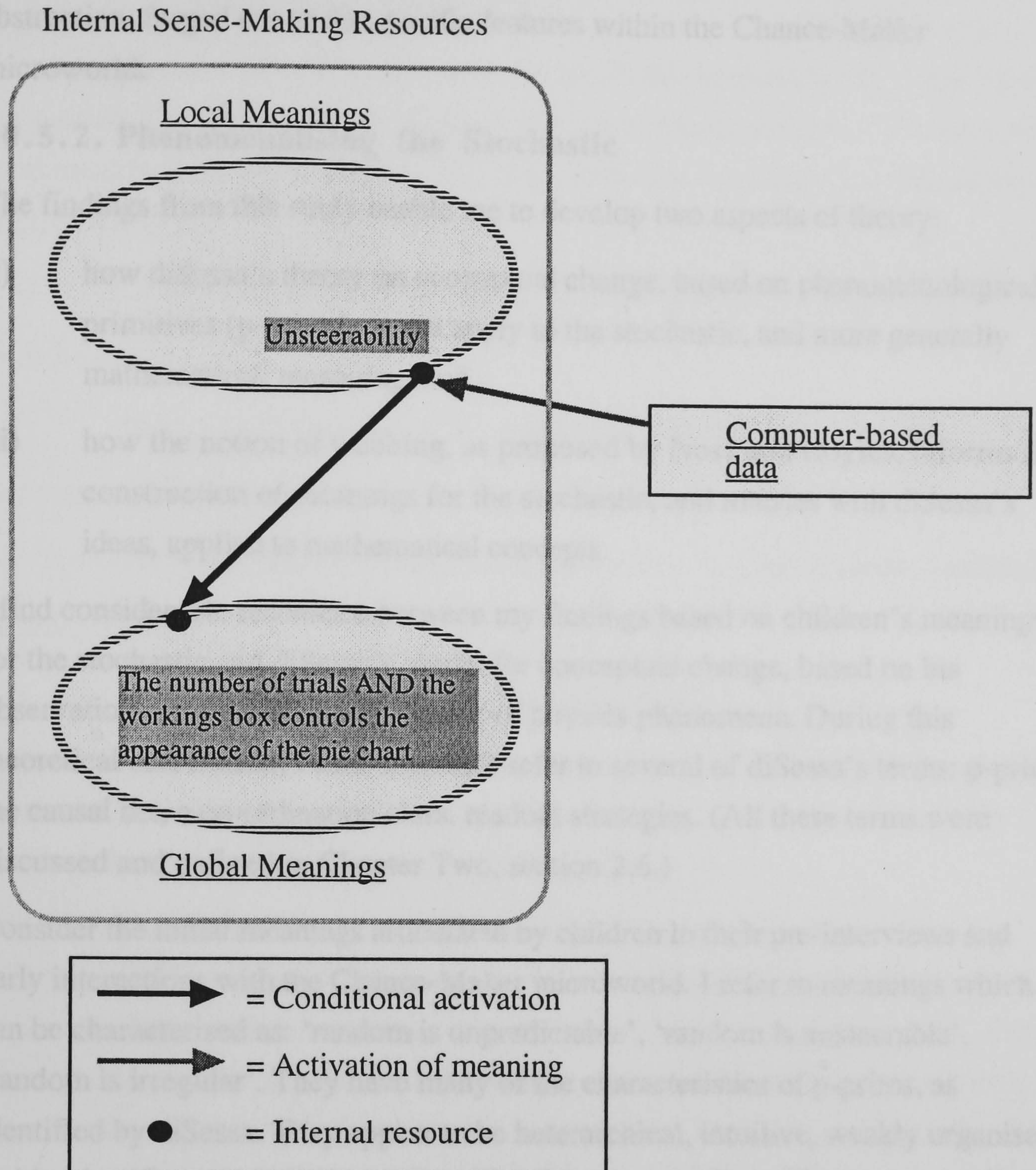

Fig. 10.1 : A tentative model for the connection of local and global meanings

Let us start with the rectangular box to the right of Figure 10.1. Surface features of the activity from the behaviour of the computer-based phenomena, as viewed by the child, activate the unsteerability meaning in the set of local meanings, part of the overall sense-making resources. The condition attached to the unsteerability meaning activates the global causal meaning; this conditional connection can be schematised as 'if <number of trials> is large then cue <connected global situated abstraction>. The connection between the local and global meanings is depicted as a heavy black arrow.

This model can be extended in similar ways to incorporate unpredictability, fairness and irregularity. In each case, the global meanings emerge out of a conditional connection between the local meaning and a corresponding cause-and-effect situated abstraction, forged out of the specific features within the Chance-Maker microworld.

10.5.2. Phenomenalising the Stochastic

The findings from this study enable me to develop two aspects of theory:

(i) how diSessa's theory on conceptual change, based on phenomenological primitives (p-prims), might apply to the stochastic, and more generally mathematical, reasoning, and

(ii) how the notion of webbing, as proposed by Noss and Hoyles, informs the construction of meanings for the stochastic, and marries with diSessa's ideas, applied to mathematical concepts.

I find considerable resonance between my findings based on children's meanings for the stochastic and diSessa's model for conceptual change, based on his observation of students, making sense of physics phenomena. During this theoretical elaboration, I shall therefore refer to several of diSessa's terms: p-prims, the causal net, a co-ordination class, readout strategies. (All these terms were discussed and defined in Chapter Two, section 2.6.)

Consider the initial meanings articulated by children in their pre-interviews and early interactions with the Chance-Maker microworld. I refer to meanings which can be characterised as: 'random is unpredictable', 'random is unsteerable', 'random is irregular'. They have many of the characteristics of p-prims, as identified by diSessa. They appear to be heterarchical, intuitive, weakly organised, lacking justificatory structure and unable to resolve conflicts. They were used by children in this study in the ways that diSessa expects p-prims to be used:

- to assess the likelihood of various events,
- to make predictions and "postdictions",
- to give causal descriptions and explanations.

There appears to be a straightforward relationship between these meanings and the sorts of experiences one might expect children to have when playing games involving coins and dice, implying that such local meanings could easily be abstracted directly from experience, which is the assumption for the initial boot-strapping of p-prims.

There is though a fundamental difference between local meanings for the stochastic and diSessa's p-prims. The local meanings for the behaviour of stochastic phenomena are NOT causal. (I am grateful to Professor Andy diSessa for his valuable comments which helped me to formulate this important distinction between the p-prims identified from his work on physics concepts and my observation of local meanings for the stochastic.) We can schematise meanings like 'random is unpredictable' as:

'if <x-attribute> holds in a situation then <y-attribute> also holds.'

or more briefly:

<property or description> ⇒ <another property or description>

In contrast, diSessa's p-prims might be schematised as:

'if <x-phenomenon> is present in a situation then <y-phenomenon> results.'

or more briefly:

<circumstance> ⇒ <result>

Both the local meanings for the stochastic and diSessa's p-prims are in the class of "reasoning", that is, trying to make sense of phenomenological behaviour. We must ask then from where this distinction might stem.

One of the functions of causal p-prims is to provide meanings, for example by deciding what is predictable under given circumstances – if we apply a force, we expect the object to move. Another function of p-prims is to help us decide what is controllable – if we vary the force, the object varies in how it moves. Prediction and control are central to causality.

What do we do when we experience situations that are not predictable or not controllable? Perhaps we set up a co-ordination class, whose task is simply to identify that which is not determined by causal factors, isolating such phenomena in a category of their own. This special class has no causal net. (The task for this class seems to be one of categorising rather than co-ordinating. Since the task for the class evolves, as we shall see, I shall nevertheless continue to refer to it as a co-ordination class.) In Piagetian terms, this is how we deal with the inability to explain random mixtures through operational thinking. We come to call this special co-ordination class, 'randomness'.

Meanings within this disjoint co-ordination class become (weakly) organised; some inferential deduction (for example, phenomena which are unpredictable are unsteerable) can now be made. Perhaps the only sign of any causal reasoning associated with this class is the meaning, 'random is fair', since this meaning is such that it can sometimes be schematised as 'fairness causes randomness' (for example, when the fairness is seen as residing in the symmetry of the random device).

Local meanings for the stochastic exist as the complement to all that is causal. Meanings for long-term aggregated behaviour can not be deduced through inference from those local meanings, nor are they likely be abstracted from direct experience of everyday phenomena.

Through interaction with the specific features of the Chance-Maker microworld, tuning towards expertise occurred. In this environment, randomness was phenomenalised by the presentation of gadgets which were explored and manipulated. Readout strategies could be used to construct new p-prims, which might in fact have been re-used existing p-prims from other co-ordination classes, extended to new domains of applicability. A causal net became properly constituted within the co-ordination class of randomness. The phenomenalisation of the stochastic allowed the use of causal p-prims in attempts to make sense of unexpected behaviour, such as the apparent predictability of the pie chart, and the evident controllability of the gadgets in the long term. In this sense, the task for the randomness co-ordination class was extended beyond categorisation ('this phenomenon is unpredictable so it is random') to incorporate cause and effect meanings ('the configuration of the workings box shapes the appearance of the pie chart').

The term, webbing, describes the construction of meanings through the forging of connections between internal and external resources. In the case of the stochastic, I propose that webbing is precisely the process of constituting or re-structuring the causal net. Prior to activity with the microworld, the children's internal resources consisted of a set of initial meanings, p'-prims, (p' to emphasise their complementary relationship to classic p-prims). The workings box presented phenomenological formalisations of randomness. Exploration of the workings box was supported by interactions with the **repeat** primitive, and use of the graphing tools. Through these tools, new p-prims were bootstrapped to create a causal net alongside the p'-prims. Causal meanings were constructed for the long-term aggregated behaviour of the gadgets. We can see the webbing of the stochastic as the construction of causal phenomenological meanings where previously such meanings were at best weak, and possibly non-existent.

In Figure 10.1, we can now interpret the set of local meanings as the co-ordination class, randomness, and the meanings within as p'-prims. The webbing process is now depicted as the construction of a conditional connection between the co-ordination class and the set of global meanings, which become part of the randomness co-ordination class. The randomness co-ordination class has been re-structured since the addition of the new connection demands a change in the domain of applicability of the p'-prims to relatively small numbers of trials and the construction of a causal net.

It is no coincidence that webbing has emerged out of the research by Hoyles and Noss of children using computer-based resources. The process of embedding mathematics into microworlds involves the phenomenalising of mathematics. It is akin to turning mathematics into physics, in the sense that children can bootstrap and make use of causal p-prims to make sense of instantiations of mathematical concepts in a computer domain, provided that other aspects of that domain support personal exploration and sense-making. By phenomenalising mathematics, it becomes accessible to both inferential (including categorical) and causal sense-making, rendering the mathematics more meaningful.

10.6. IMPLICATIONS FOR FUTURE RESEARCH

Much of this last section must be regarded as conjecture, especially when the limitations of the research are taken into account. If these conjectures were valid, then we might expect that similar findings would result from related research.

All the children studied were taken from a narrow age range. The reasons for the choice of age have been discussed in Chapter Four; further research, undertaken initially with adjacent age ranges, would help to test the validity of the theoretical framework, and begin to focus on developmental issues. A longitudinal study might more meaningfully analyse the development of meanings across age ranges, and overcome a limitation of this study — the amount of time spent with each child or pair of children.

There is a neat consistency in the unification of diSessa's theory of conceptual change and the notion of webbing. Research is needed to explore these ideas further. The theoretical sketch is not entirely limited to the stochastic, and a particularly productive direction for future research may be to research other mathematical domains. Below I outline the rationale for such research.

Suppose randomness was not the only co-ordination class with a weak or non-existent causal net. There is reason to believe that this may be so. Mathematics at its highest levels tends to derive its meanings through inference, in which theorems are deduced from prior theorems or axioms, with no reference to phenomenological behaviour, which would mitigate against rigour. There has been a tendency for instruction to re-play the nature of mathematics as a discipline. That is to say, mathematics teaching has tended to present the mathematics concepts in their de-contextualised forms, compelling children to try to make sense of these concepts by inference, or to avoid sense-making and to adopt instead a rote-learning strategy.

It is not unreasonable to conjecture that under such circumstances there may be many mathematical co-ordination classes which have weak of non-existent causal nets, lacking therefore intuitive meaning constructed through causal p-prims. Perhaps such co-ordination classes would be rendered more meaningful if they were phenomenalised, following similar principles as those set out for the phenomenalisation of the stochastic in this study. At the end of the next section, I outline one example.

10.7. IMPLICATIONS FOR TEACHING AND LEARNING

The 1960's, through to the 1980's, has seen the rise in the UK, now in sharp decline, of investigational approaches towards the teaching and learning of mathematics. At the same time, there has been increased use of physical apparatus to support the learning of new concepts. As a result, it is possibly fair to say that mathematics has been perceived as more fun, though not necessarily as more effective from a learning perspective. Perhaps the emphasis on investigation has not been matched by equal care with regards to the tools available for that exploration.

In his analysis of intuitions of the stochastic, Fischbein suggested that more experimentation with, and more attention in schools towards, the stochastic might redress the imbalance towards the deterministic, and enable early intuitions of relative frequency to flourish.

Whilst I have some sympathy for the general direction of Fischbein's arguments, like the general investigational movement, he perhaps is too vague about the nature of that activity.

If experimentation in the classroom does little more than what can be done in everyday activity, what reason is there to suppose that the children will abstract new meanings? The stochastic pervades our everyday activity yet this study shows that the meanings that we abstract are limited to those that are abstracted directly from short-term behaviour.

Along the lines of the design of the Chance-Maker microworld, such activity must incorporate:

- the initial meanings of unpredictability, unsteerability, irregularity and fairness,
- a purpose for the activity in which it makes transparent sense for the children to use specially provided tools, designed to optimise the construction of new global meanings for long-term stochastic behaviour ,
- a facility to amend formalisms of probability distribution, in order to explore personal conjectures for stochastic behaviour,
- a facility to repeat easily a large number of trials and replicate experiments,
- a facility for inspecting lists of results,
- graphing tools which support the search for patterns in the aggregation of results.

Such tools have the effect of phenomenalising the stochastic, perhaps enabling children to use both their cause-and-effect and inferential internal apparatus to make sense of the long-term aggregated behaviour of the stochastic.

As a concluding thought, we might consider how similar principles might be applied to phenomenalising other areas of the mathematics curriculum.

For example, what might the phenomenalisation of equation solving look like? In fact, teachers have for many years tried, with limited success, to offer metaphors of balance to help children to solve equations. In so doing, these teachers are intuitively attempting to phenomenalise the solving of equations (indeed the notion of balance has been shown by diSessa to be part of our intuitive sense of mechanism). A microworld, designed on the same principles as the Chance-Maker microworld, might offer children direct manipulation of culturally-based simulations — a seesaw comes to mind as one example. The seesaw would incorporate a formalism, a situated representation of an equation, in such a way that the children would move towards control of the seesaw through this formalism. Through phenomenalising equation solving, children might construct situated abstractions, which correspond to the familiar 'rules' for the manipulation of equations but which have, in this environment, a utility related to control of the seesaw, rendering the act of equation solving as intuitively meaningful.

I offer the above brief example to outline how the meanings constructed for the Chance-Maker microworld might be applied to the phenomenalisation of other mathematical domains, in which children might construct new causal meanings.

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APPENDICES

A1. ITERATION 1

A1.1 Schedule for semi-structured pre-interviews

SNAKES & LADDERS — OFF THE COMPUTER

Q1. Do you play board games? ..... Which board games do you like?

Q1a When you play a game like ....*(choose a game which depends mostly on skill)*, do you usually win or ....Why is that?

Q1b Do you know the game 'Snakes & Ladders'. When you play this game, do you usually win or .... Why is that?

Q2. Suppose you were in this situation in a game of Snakes & Ladders.

Have a Snakes & Ladders board nearby for quick reference. The situation shows a snake which is exactly six squares away from the counter and a ladder which is exactly one square away from the counter.

What might happen?......Which do you think would happen? ......Which is most likely to happen?

Q3. Now in this game of Snakes & Ladders, we are using two dice. We decide how far to move by adding the scores together. Suppose you were in this situation in this type of game of Snakes & Ladders.

The situation shows a snake which is exactly nine squares away from the counter and a ladder which is exactly five squares away from the counter.

What might happen?......Which do you think would happen? ......Which is most likely to happen?

Q4. Now, in this game of Snakes & Ladders, you get to choose where to put the snake. We are playing with one dice only again. Here's your counter, on square 2. You have to put the snake somewhere within six squares. In other words, you have to put your snake on square 3, 4, through to square 8.

Where are you going to put it? ...... Why is that?

Q5. Now, we are playing with two dice again, adding them up. You have to choose where to put the snake. Here's your counter, on square 2. You have to put the snake somewhere within twelve squares. In other words, you have to put your snake on square 3, 4, through to square 14.

Where are you going to put it? ...... Why is that?

ROLL-A-PENNY — OFF THE COMPUTER

Q6. Imagine rolling a pound coin down this slope. Show me how far you think they would go ....Would they all go the same distance?

Q7. Look at this diagram *(Figure A1)*. I've drawn some lines to mark off distances from the slope. Suppose that the first coin I roll finishes up here *(mark a cross to show where the coin finishes)*. Mark where you think the next twenty coins might finish up.

Q8. I want you to roll these coins down the slope. Each time I want you to try to predict where you think the coin will finish up. Then you can mark it's finishing position with these stickers.

The subject is given a pound coin and the Roll-A-Penny apparatus (Figure A2).

Are there any patterns in the stickers?

Q9a *If "yes", I will ask them to describe in words the pattern:* "Can you explain why you get this pattern?

Q9b *If "no", I will ask,:* "Can you explain why there is no pattern?

Q10 Where is the most likely place for the next coin to finish up?

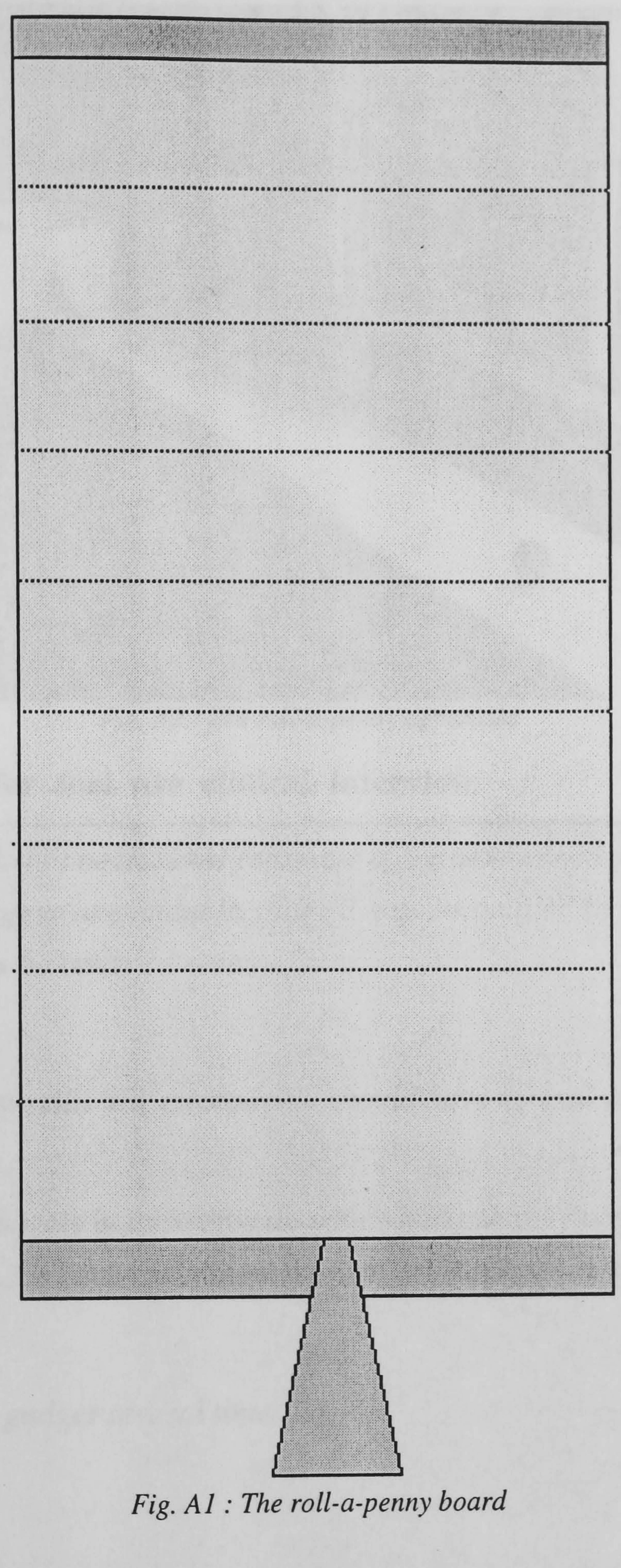

Fig. A1 : The roll-a-penny board

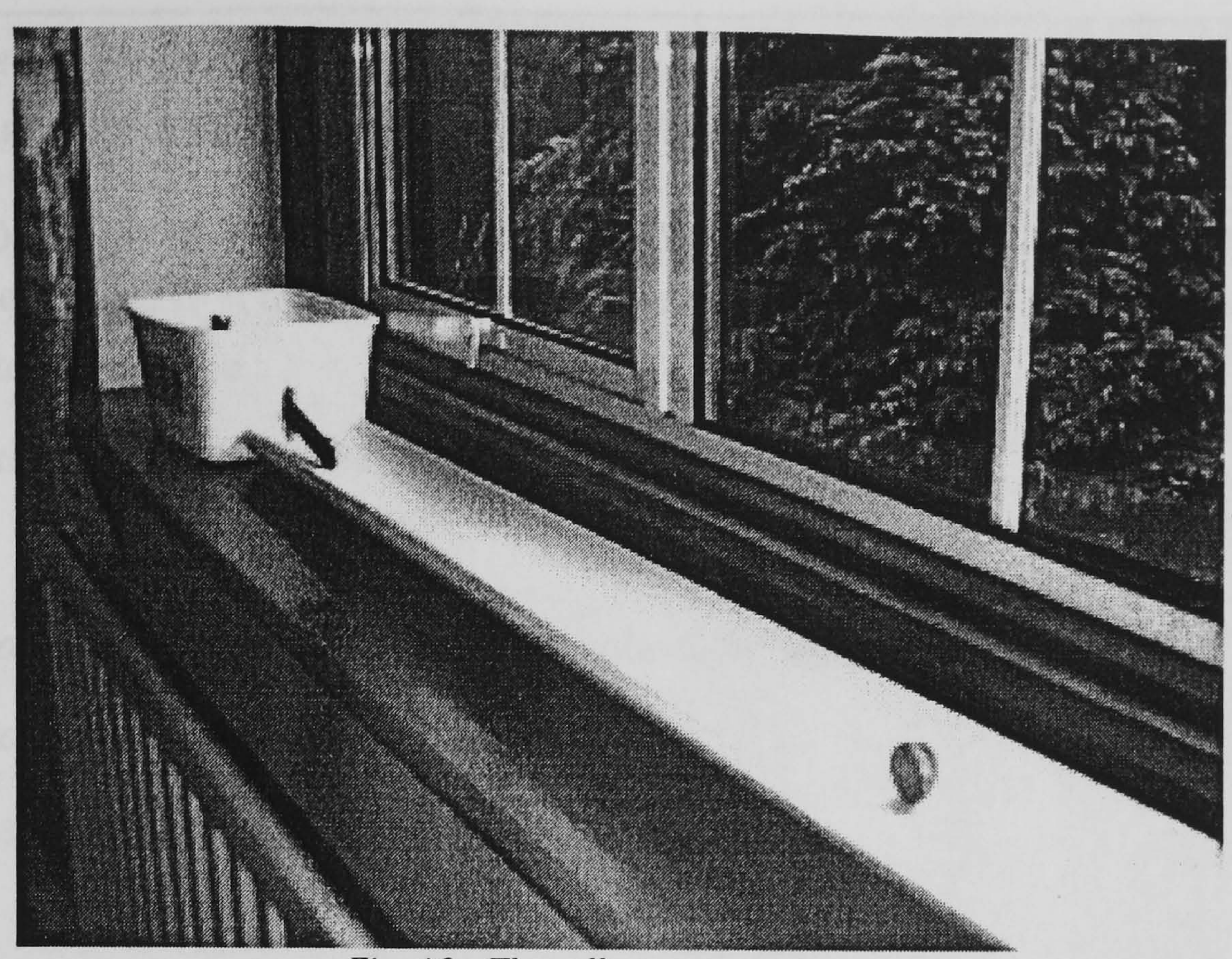

Fig. A2 : The roll-a-penny apparatus

A1.2 Schedule for tool use clinical interview

The stages below do not necessarily represent a pre-determined progression, though the first stage is unavoidable. Stage 3 may be omitted by some subjects. Stages 2 and 3 may be revisited several times.

STAGE 1

Open up the microworld . The microworld should have the roll-a-penny gadget closed right down.

There are lots of gadgets in this software which you can play with and use.(*Click on the dice gadget.)* What do you think this gadget is supposed to be?

.....

Click on the clock gadget several time....

And this one?

....

You can find out how these gadgets work by opening them up. You do this by double clicking.

Double click on dice gadget to show the idea — but do not enter into discussion about what is there.

You can even change what the gadget does.

Change the dice so that it chooses from 1-9, close the gadget down, show it working, open up dice again, change it back to 1-6, close it down again.

I want you to play with these gadgets and try to decide what they are doing and how they work. If you do not understand any of them, don't worry because I'm interested in that too.

Leave them to play with the gadgets.

....

After some time, enter into a discussion about the gadgets.

Point to the dice gadget.

What is this gadget supposed to be?

....

How does it work?

Do you think it works well?

How do you know it works (doesn't work) well?

....

Point to other gadgets and repeat

....

After these discussions ....

OK — now there is another gadget which is meant to work like the Roll-A-Penny game that we played last time.

Open up the Roll-A-Penny gadget.

You operate it just like the other gadgets.

Click on the board.

It shows you how it works down here, so you don't need to open it up. You can change how it works just like the other gadgets.

....

After some time, discuss the Roll-A-Penny gadget.

How does it work?

Do you think it works well?

How do you know it works (doesn't work) well?

....

If the subject believes that this gadget is working well — perhaps because it is perfect — ask them:

Did the pound coins roll exactly the same distance every time?

and / or

This gadget is supposed to be like the one you used. Was that a perfect gadget?

....

OK, so this gadget is not working very well, is it? Your task is to mend it. You will need to change how it works.

STAGE 2

The subject may see the pattern in what the coin does in a deterministic way, in which case he will need to know how to make a pre-determined sequence of moves (e.g. using the clock gadget). They may need some help on the technicalities but try to minimise the amount of help offered.

Be prepared to intervene with:

In the real Roll-A-Penny situation, do you know, before rolling the coin, where it will finish up?

The subject may see the pattern as stochastic, in which case they will use other gadgets, such as dice or ....I may need to give help on the technicalities but try to minimise the amount of help offered.

Be prepared to intervene with:

What sort of patterns did we get when we rolled the real coins?

and / or

Were the stickers spread out evenly?

STAGE 3

At this point the subject may wish to return to the real situation and do some further analysis. The question is whether the subject now looks at the real situation with a new perspective and whether this new perspective shows in the way he continues the simulation on the computer.

Would you like to play with the real coins again?

and / or

How far do the pound coins tend to roll?

and / or

Are the pound coins evenly spread?

AUXILIARY QUESTION

Your task is to make the computer's roll-a-penny behave like the real thing. Would it help to collect some data on the real thing?

A2. ITERATION 2

A2.1 Schedule for semi-structured pre-interviews

SINGLE DICE

1.1 What is this (show a dice) ?

1.2 Where might you use a dice?

1.3 Why is a dice useful in those games?

1.4 If you did not have a dice, what else could you use?

1.5 How do you know that a dice is random?

1.6 Do you know any board games where you need to throw a six? .... What do you do when you are in that situation? .... Does your method work?

1.7 Do you think a six is harder to get, just the same, or easier to get than say a three? Is there one number which is harder than the others? .....Is there one number which is easier than the others?....Why is that?

1.8 Do you think this dice is working properly (hand over the dice) ?....How do you know?

TWO DICE

2.1 Here are two dice which I know are working properly (hand them over). If we throw the two dice together and add up the two scores, what could we get?

2.2 Is there one total which is harder to get than the others? Is there one total which is easier to get then the others?......Why is that?

2.3 Do you know any games where it is good to throw a double?....What do you do when you are in that situation?.......Does your method work?

2.4 Do you think a double is harder to get, just the same, or easier to get than a total of 3? ....Why is that? .....Do you think a double is harder to get, just the same, or easier to get than a total of 7? ....Why is that? ......Do you think a double is harder to get, just the same, or easier to get than a total of 5? ....Why is that?

ROLL-A-PENNY

3.1. I am going to roll a pound coin down this slope (show them the roll-a-penny apparatus, and then roll a coin). Now, I am going to roll another coin. Do you think it will go less distance, the same distance, or further?......Why is that?

3.2 OK, so it went (less, same, further>. Why did it do that?

3.3 Do you think you could make it roll the same distance?.....How?

3.4 I want you to roll this coins lots of times down the slope. Try to make it go the same distance every time. You can mark where it goes with a sticker each time. When you have done it lots of times, I want you to describe what happened.

3.5 Would you say the distance the coin rolls is random?....Why is that?

OTHER PHENOMENA

I am going to ask you now about a number of other situations. In each case, I want you to tell me whether you think the situation is random or not, ... and why.

The order of the questions will be randomised.

4.1a You toss a coin to see whether it is a HEAD or a TAIL.

4.1b You watch a person you do not know tossing a coin. You wonder whether it will be a HEAD or a TAIL.

4.2a You pull a ball from a bag. The bag has four red balls and one yellow ball in it. You do not look at the balls. You wonder whether it will be red or yellow.

4.2b You watch someone you do not know pulling a ball from a bag. The bag has four red balls and one yellow ball in it. The person is not looking at the balls. You wonder whether it will be red or yellow.

4.3a You are about to run 100m and you wonder how long it will take you.

4.3b You watch sprinter about to run 100m and you wonder how long it will take her.

4.4a You shuffle a pack of cards and then you look at the top card to see what suit it is.

4.4b You watch someone shuffling a pack of cards. They are about to turn over the top card. You wonder what suit it will be.

4.5a As a growing child, you are interested in what height you will be when you grow up.

4.5b You look at a child the same age as you. You do not know her, but you wonder what height she will be when she grows up.

4.6b As a football supporter you wonder whether Manchester United will beat Leeds United next time they play each other.

4.6a You wonder whether you will beat your friend at table football next time you play him/ her.

A2.2 Schedule for tool use clinical interviews

I am programming these gadgets to make them behave like the real world. So, for example, I want the coin to behave as far as possible like a real coin. Play with the coin and tell me what you think.

After a few minutes in which the subjects familiarise themselves with the top-level controls, use the coin gadget to introduce the tools. I intend to discuss the controls, the information box, workings box, the results box, and graphing.

Use these tools to decide whether the coin is behaving like a real coin.

Allow the subjects to play with the coin gadget and to move onto other gadgets in any order. For any particular gadget, questions should focus on the following areas of interest:

Q1 Can you control the gadget?

Q2 What would happen next?

Q3 Is the gadget random? What do you mean? How do you know?

Q4 Do these gadgets behave as in the real world?

Q5 Does the strength affect the result? Why?

Q6 Why did you throw the gadget ... times?

Q6a Why did you not throw it ... times (a number less than that chosen)?

Q6b Why did you not throw it ... times (a number more than that chosen)?

Q6c What do you expect to happen if you throw it another ... times (the same number as that chosen)?

Q7 What does this command **choose-item** mean? How does the computer choose?

Q8 Do the results in the results box help you to decide if the gadget is random?

Q9 What would the results look like if the gadget were NOT random?

Q10 What does the workings box mean?

Q11 What do these results tell you?

Q12 What does this pictogram tell you?

A3. ITERATION 3

A3.1. Schedule for semi-structured pre-interviews

SINGLE DICE

1.1 Where might you use a dice like this?

1.3 Why is a dice useful in those games?

1.4 If you did not have a dice, what else could you use?

1.5 What might happen when I throw this dice?

1.6 Do you think that the result of throwing a dice is random?

1.7 Do you know any board games where you need to throw a six? .... What do you do when you are in that situation? .... Does your method work?

1.8 Can you control the result of throwing a dice?

1.9 Do you think a six is harder to get, just the same, or easier to get than say a three? .....Is there one number which is harder than the others? .....Is there one number which is easier than the others?....Why is that?

1.10 A friend tells you that this dice is unfair. How would you go about deciding whether your friend is right?

SPINNER

2.1 What might happen when we spin this spinner?

2.2 Do you think the result of spinning this spinner is random?

2.3 Can you control the result of spinning the spinner?

2.4 Is there one number which is harder to get on this spinner? Is there one number which is easier to get?

2.5-8 What about this spinner? *Repeat above questions.*

TWO DICE

2.1 If we throw the two dice together and add up the two scores, what might happen?

2.2 Is there one total which is harder to get than the others? Is there one total which is easier to get then the others?......Why is that?

2.3 Do you think a double is harder to get, just the same, or easier to get than a total of 3? ....Why is that? .....Do you think a double is harder to get, just the same, or easier to get than a total of 7? ....Why is that? ......Do you think a double is harder to get, just the same, or easier to get than a total of 5? ....Why is that?

FRISBEE AND ROLL-A-PENNY

3.1 Suppose I throw this Frisbee and measure how far it travels. Now suppose I do that again. Do you think the Frisbee would travel exactly the same distance? Why is that?

3.2 Suppose I throw the Frisbee with exactly the same strength. Do you think it would travel the same distance?

3.3 Can you control how far the Frisbee goes?

3.4 Is the distance travelled by the Frisbee random?

3.5-3.8 *Repeat for Roll-A-Penny*

OTHER PHENOMENA

I am going to ask you now about a number of other situations. In each case, I want you to tell me whether you think the result is random or not, ... and why.

4.1 You toss a coin to see whether it is a HEAD or a TAIL.

4.2 You pull a ball from a bag. The bag has four red balls and one yellow ball in it. You do not look at the balls. You wonder whether it will be red or yellow.

4.3 You are about to run 100m and you wonder how long it will take you.

4.4 You shuffle a pack of cards and then you look at the top card to see what suit it is.

4.5 As a growing child, you are interested in what height you will be when you grow up.

4.6 As a football supporter you wonder whether Manchester United will beat Leeds United next time they play each other.

4.7 You wonder whether you will beat your friend at table football next time you play him/ her.

A3.2. Transcripts of Anne's and Rebecca's pre-interviews

Child's responses are in bold.

I am Anne and I was born on 17th January.

And what was the year?

I think it was.......

Are you a Year 6 or a Year 5?

Year 5.

You're a Year 5. Ok. We're going to start with the dice. Where might you use a dice like that.

For a board game.

And why are dice useful in games like that?

Because you can roll them and move spaces.

If you didn't have a dice. If the dice had been lost, what else could you use? Is there anything you could perhaps make?

You could make the dice.

Is there anything different from a dice that you could use or make?

Not really if it's a board game and you need the numbers and you need to roll something.

Ok. What might happen when I throw this dice? What could happen?

It will land on a different number.

What numbers could it land on?

1 up to 6.

1 up to 6, Ok. Do you think that the result of throwing a dice is random? Is random a word that you've come across before? Have you heard it at all before?

No.

Ok, you can't answer then I suppose. Do you know any board games where you need to throw a 6?

Yes, it's called Frustration.

Is it the sort of game where you need to throw a six to get started?

Yes.

So, if you're in that situation, is there anything you would do to give yourself a better chance of getting a six?

Not really.

Not really no. Is it possible to control the result of throwing the dice do you think?

Yes, you could throw it gently.

And if you don't throw it gently, if you throw it hard, can you influence what number comes out?

No.

No. Is there any number on a dice which is harder to get than any of the others?

Not really.

Is there any number on a dice that is easier to get?

No.

No. Why is that? Why are they all the same?

Don't know.

Ok, supposing a friend has been using this dice and they tell you its unfair and they leave the room and leave it with you. What would you do to try and decide whether its unfair or not?

Roll it.

Ok, and how would that tell you?

Because it would probably come up the same number every time.

So even if it didn't come up the same number every time, would it still be unfair?

No.

You think not. Is there anything else you can do to decide whether the dice is unfair?

No.

How many times would you roll it, you said you would roll it a few times, how many times would you roll it?

About three.

Fine. I want to now change to looking at these spinners. Start with this one. What might happen when I spin this spinner.

It will come to number 5 again, because you started it with number

5.

What could happen, what are the other possibilities?

You could have spinned it harder and it could have gone a bit further.

Ok, do you think that the result of spinning this spinner is random?

(Unintelligible — too quiet)

Can I control or can you control the result of this spinner?

No.

No, you can't control it. Ok. Would you say that there's one number on here that's harder to get than the others.

No.

Would you say that there's one number that's easier to get than the others?

Yes, five.

Number five is easier. Why do you think that?

Well, because when you spinned it the first time, it came to number five again.

Ok, and why do you say that there's no number that's harder to get than the others?

Because you can't really know what number's going to turn up.

Ok, I want to come now to this spinner here. What might happen when I spin this?

It might go to number one, because it's much bigger than the squares.

So number one is much bigger, so you think it will go on number one. What else could happen?

It might go past it.

Ok, I'm not sure there's any point in me asking this, but I think I'll try. Do you think that the results of spinning this spinner is random?

Don't know.

You don't know. All right, I'll stop asking you that sort of question. Can you control the result of spinning this spinner?

No.

No, Ok, is there one number that is harder to get than the others?

Yes, I think it might be number four because it's much smaller than the others.

I think you've already answered this, but I'll ask anyway, is there a number that's easier to get than the others?

Yes, number one.

And why was that?

Because it's much bigger than the other numbers.

Ok, that's fine. So if we throw the two dice together, like so, and look at what we get and we add up the two scores, so I'm interested in totals. Ok, different totals. What might happen?

You'll get a different number because it goes up to six so you'll get a different number because it will be much higher than the other numbers.

And what could the total be? If I shake these two dice, what different totals could I get?

Different highnesses.

Different highnesses, different totals. What's the biggest total I can get?

Twelve.

Twelve, and how would you get that?

Two sixes.

Ok, and what's the lowest total I could get.

Two.

Two and that would be a double one.

Yes.

Can you get every other total in-between the two and the twelve. Is it possible to get every other total in-between. Is there one total out of all of those that is harder to get than the others.

No.

Is there any total that's easier to get than any others?

No.

No, and why is that.

Because you can't estimate what number you'll get because they're all fair, both the numbers are fair.

Ok. I want to talk about doubles now. Like it's showing a double four. Ok. So suppose you're playing a game, like monopoly for instance when you have to get out of jail and you need to shake a double. But like in Monopoly it doesn't matter what double, it could be any double that gets you out of jail. And I want you to compare that to throwing other things, such as a total of three, a two and a one for instance. I want you to try and decide, do you think, throwing a double is harder, just the same or easier to get than throwing a total of three?

Harder.

You think throwing a double is harder. Can you explain why?

Because if you throw two dices it normally comes out a different number, it doesn't normally come out a double number.

Ok, so you think throwing a double is harder. And what about a double, is that harder, just the same or easier to get than a total of seven?

Harder.

You think the double's harder, can you explain why?

Because you need two different numbers, to get a second set would be much harder.

Ok, and what about a double compared to a total of five. Is the double harder, the same or easier to get than the total of five?

Harder.

The double's harder and is that the same reason again?

Yes.

Now I want to talk to you about Frisbees, you know what a Frisbee is.

Yes.

Can you briefly tell me what a Frisbee is.

It's a sort of circle and you throw it to each other, spinning.

Suppose that you're throwing the Frisbee and you measure how far it goes when you throw it. And you're about to do that again and again you'll measure it. Do you think it will measure the same distance as last time?

Probably not.

Ok, can you explain why?

Well, because you could throw it harder, or the wind could take it further.

Ok. So, suppose you threw it with exactly the same strength as last time, would it go the same distance then?

No.

Because of the wind that you just mentioned for instance?

Yes.

Suppose I throw the Frisbee....... no forget that. Can you control how far the Frisbee goes?

No, because of the wind again, it could take it away if a really strong wind just came past.

No. Would you say the distance travelled by the Frisbee is random. You don't know do you?

No.

I want to ask some similar questions now about the roll-a-penny which is over there. Except it's a roll-a pound coin because it's pound coin I've got here. What I'm going to do is put in this chute at the end here right next to the red tape, and just let go. You can see how far it goes, supposing I do that again, put it next to the tape and just let go, do you think it will go the same distance.

No.

Probably not. Why is that?

Because *(unintelligible — too quiet)*

I think you're right, I don't think it will go the same distance, I wonder why?

I just don't think it will.

Ok, the last set of questions in each case, I was going to ask you whether you thought they were random or not, but I don't think I'm going to get any answer from you because you clearly haven't come across the word before, so I think we'll leave those questions, so the we can finish in time for break. Ok then, thanks ever so much and perhaps later in the week we'll get the chance to work on the computer. Thanks very much.

Rebecca

The tape recorder failed to operate correctly in the original interview. I was able to call Rebecca back immediately and conduct a second interview in which we reviewed together Rebecca's responses. This is the transcript of that review.

I'm Rebecca and my date of birth is the 11th July 1986.

So we talked about a dice and I asked you whether the result of throwing a dice is random. Could you tell me again what you think.

Well it's random cos you can't control which number it goes on.

And then we talked about board games in which you need to throw a six and I asked you whether there's anything you could do to give yourself a better chance of throwing a six. What do you think?

No, there's no way.

Ok. Can you control the result of throwing a dice?

No.

Are there any numbers on a dice which are harder to get than any others?

Not really.

And are there any that are easier to get than any others?

No.

And can you tell me again why that was.

Because it's random.

And if a friend told you that this dice is unfair, what could you do to test that out?

You could roll it a few times, maybe ten or twenty times and if you get the same number, or one number more times than others.

Ok, then we talked about these spinners. I asked you, do you think the result of spinning this spinner is random?

Yes.

Can you explain why?

Cos there's the same amount of space for each number.

Ok. Can you control the result of spinning this spinner?

Only if you spin it slowly.

But if I spin it fast?

No, you can't.

And what about this other spinner. Is the result of spinning that spinner random?

No.

Can you explain why?

Cos there's more space for one and less space for four.

So would you say there's a number that's easier to get than the others.

One.

And a number that's harder to get?

Four.

And therefore you're saying that that's not random. And then we talked about throwing two dice. What we're doing is throwing the two dice and looking at the total we get, and in this case we got a four, and I asked you, is there one total which is harder to get than the others?

No.

Is there one total which is easier to get than the others?

No.

And can you explain why?

Cos it's random, you can't control which number it lands on.

Ok, and then we talked about doubles, like a double five and then we compared getting a double to other situations. So, is scoring a double harder or just the same or easier to get than a total of three?

It's random so you can't control it.

So is it harder, just the same or easier to get.....

It's just the same.

And then we talked about comparing a double to other things, like a total of seven or a total of five and in each case I think you said the same thing. Just tell me again what it is in each case.

It's random and you can't control it.

So, it's just the same.

Yes.

Then we talked about a Frisbee and you were able to tell me what a Frisbee was and I asked you if you throw a Frisbee and measure how far it goes and then throw a Frisbee again do you think that the Frisbee will travel exactly the same distance?

No.

Why is that?

Because there's other factors like the wind.

And even if you throw it with the same strength, the wind could cause it to go a different distance.

Yes.

So, can you control how far the Frisbee goes?

It depends how far you throw it, if you throw it lightly it won't go as far as if you throw it hard.

So, you can control it.

To an extent.

Is the distance travelled by a Frisbee random?

Yes.

And why is that?

Because you can't really control what happens.

So you can control it to an extent, but not completely.

Yes.

Then we asked some similar questions about the roll-a-penny situation, and we rolled a coin down and then we asked you, if we do it again will it go the same distance?

No, because it might not go the same distance, it depends on the friction.

And would you say that the distance the roll-a-penny goes is random?

Not really.

Why is that.

Well it might go the same distance but it might not, so it's in-between.

Then I asked you about some other situations, you toss a coin to see whether it's a head or a tail, is that random?

Yes, because you can't really control it.

You pull a ball from a bag which has got four red balls and one yellow ball in it. You don't look at the balls, you wonder whether it will be a red or a yellow. Is that random?

Sort of because there's more reds than yellows, so there's not much chance of you pulling out a yellow, there's more chance or you pulling out a red.

And so, what are you saying, is it random or not?

Well, it's sort of in-between.

Ok, you're going to run the 100 metres and you wonder how long it will take you. Is the time that it takes you random?

Yes. Because you might go faster to try and beat the last time.

Ok, you shuffle a pack of cards and you look at the top card to see what suit it is. Is that random?

Yes, cos you don't know what you're going to pick off the top.

As a growing child, you're interested in what height you'll be when you grow up. Is that random?

Well you grow at a fairly steady rate, so it's not random.

You wonder whether Manchester Utd will beat Leeds Utd next time they play. Is that random?

Yes. Because the teams might have had different training.

And you wonder whether you'll beat your friend next time you play her at table football.

Yes, because it depends on your skill and her skill.

Ok, and then finally, I asked you to summarise looking back over all the questions, what it means to say something is random. How do you decide whether something is random.

Well you can't control something that is random, so it could be anything, anything could happen, so if you're throwing a dice it could be any number.

Ok., we've done a quick review of what we did last time and I think on the questions you've answered the same things. Thanks very much.

A3.3. Schedule for tool use clinical interviews

STAGE 1 : UNSTEERABILITY

I am programming these gadgets to make them behave like the real world. So, for example, I want the coin to behave as far as possible like a real coin. Play with the coin and tell me what you think.

It is my intention that this play will be entirely at top level, as I have not yet introduced the various tools and structures available inside the gadgets.

1.1 Could you control the coin?

1.2 Could you predict what would happen next?

1.3 Is the coin random?

I am going to ask you to play with the spinner, the Frisbee and the roll-a-penny.

1.4-6 Do you think you can control the spinner/ Frisbee/ roll-a-penny?

1.7-9 Will you be able to predict what will happen next?

1.10-12 Is the spinner/ Frisbee/ roll-a-penny random?

Play with each of these gadgets now and decide whether you were right in each case.

1.13-15 Could you control the spinner/ Frisbee/ roll-a-penny?

1.16-18 Were you able to predict what would happen next?

1.19-1.21 Is the spinner/ Frisbee/ roll-a-penny random?

STAGE 2 : DETERMINISM

You have played with the coin, the spinner, the roll-a-penny, the Frisbee and the roll-a-penny.

2.1 Do these gadgets behave like in the real world?

2.2 Would a real Frisbee travel different distances for the same strength? Why?

2.3 What about the roll-a-penny?

2.4 Would a real coin give different outcomes, heads and tails, even if the strength was the same? Why?

2.5 Would a real spinner give different outcomes, different numbers, even if the strength was the same? Why?

STAGE 3 : OPENING UP

I am now going to open up the coin gadget to show some tools that you can use to investigate the coin further.

I intend to discuss the controls, the information box, workings box, the results box, and graphing.

Use these tools to decide whether the coin is behaving like a real coin. Is the coin random?

Subsequently, I intend to ask them to investigate in a similar way the following gadgets: SPINNER, DICE

I will take opportunities that arise to probe into the following issues.

3.1 Why did you throw the gadget ... times?

3.2 Why did you not throw it ... times (a number less than that chosen)?

3.3 Why did you not throw it ... times (a number more than that chosen)?

3.4 What do you expect to happen if you throw it another ... times (the same number as that chosen)?

3.5 What do you mean when you say it is random?

3.6 What does this command CHOOSE mean? How does the computer choose?

3.7 Do the results in the results box help you to decide if the gadget is random?

3.8 What would the results look like if the gadget were NOT random?

3.9 What does the workings box mean?

3.10 What do these results tell you?

3.11 What does this pictogram tell you?

3.12 What does this pie chart tell you?

3.13 Why did you choose the pictogram/pie rather than the pie/pictogram?

3.14 How do you know whether the gadget is behaving randomly?

STAGE 4 : DICE EXPERIMENTS

4.1 Can you make me a dice which is NOT random?

4.2 Can you make me a dice which makes it harder to get a six than any of the other numbers? All the other numbers are equally easy.

4.3 I want you to throw a fair dice 10 times and look at the pie chart. Then compare it with what you get when you throw the dice 100 times. Do you notice any difference. Why is this?

STAGE 5 : TWO-SPINNER EXPERIMENTS

5.1 I have two spinners here, each marked 1, 2 and 3. If I throw the spinners what totals can I get? Are each of these totals just as easy to get? Is one total harder to get than another?

5.2 Look at the two-spinners gadget. What does the workings box mean?

5.3 Can you make the two-spinners work like the real spinners?

STAGE 6 : TWO-DICE EXPERIMENT

6.1 Here is a two-dice gadget. The result is the total of the two dice. Have a look inside and tell me whether you think it will behave like two real dice.

6.2 What would you need to do to make it work properly?

NOTE : THE OTHER INTERVIEW TRANSCRIPTS CAN BE FOUND ON THE WORLD WIDE WEB AT ADDRESS:

http://www.warwick.ac.uk/wie/staff/DP.htm

A3.4. Case Account of Anne's and Rebecca's clinical interview

The coded references used down the left-hand side of the case account are used throughout the main body of the thesis. Anne and Rebecca was originally transcribed and analysed as the sixth case out of the eight in Iteration 3 (this is not meant to imply that it was the sixth of the eight interviews in chronological order). Hence all these references begin with a 6.

6.1. *Introducing the gadgets*

6.1.1. (Time [0:00:00]) Anne and Rebecca have not used Boxer before. Anne and Rebecca listen while I explain that I am trying to program these gadgets to make them behave as much as possible like their real world equivalents, but they may decide that some gadgets need mending.

6.1.2. I demonstrate the way that the strength control can be used to flip the coin and how they can replicate experiments with exactly the same strength.

6.2. *Coin*

6.2.1. (Time [0:03:40]) They being to play with the coin. After an initial click with 70% strength, which is a tail, they decide to replicate this experiment by clicking the gadget itself. Anne says, "to see if it comes up tails again." In fact it is a head. Then 100% strength gives a head, and then head again, and then head again. Anne says, "It's a bit unfair because it keeps going to the queen *(i.e. heads)*." They try 100% again and it gives a tail, and then head. Anne says, "Most of the time it goes to a queen, er heads." They now try low strengths. A 40% strength gives tails and then tails again.

6.2.2. I ask Anne if she thinks she can control the outcome on the coin. Anne: "No." Rebecca thinks not as well. Neither think they can predict the outcome. I ask if the coin is random. Rebecca: "Not really ... it's probably been programmed to do it, in a loop." Anne says she doesn't know. I ask Rebecca what she meant by 'in a loop'. Rebecca: "Well, it's programmed to do heads, then maybe heads again and then tails." I clarify, "In some sort of pattern?" Rebecca: "Yes." I ask if she was able to pick up some sort of pattern. Rebecca: "Not really."

6.3. *Spinner*

6.3.1. (Time [00:07:12]) They being to play with the spinner. They try 100% strength and it gives 1. 100% strength again gives 4, then 5, then 1. 60% strength gives 1 and then 70% strength gives 1 again. Anne says, "I think it might be a bit unfair because the 1 is much bigger than the others .... The 2, 3, 4 and 5 is much smaller than the 1 so 1 has more of a chance."

6.3.2. Strength 10 gives a 1. 90 gives 1. Rebecca agrees with Anne. I ask if they think they can control the outcome. Anne: "No, well, sort of ....It normally lands on 1. The slower you go it seems to land on 1 as well." I ask if that means you can control it or not. Anne says, "Maybe not." Rebecca: "I don't think you can really control it, because if you do it slowly, you never now, it could come on a 2, if it starts off on a different number." They don't think you can predict the next result. I ask if they think it is random. Rebecca: "I think the same again. It's probably programmed." Anne says she doesn't know. I say, "You're nor sure about this word, are you?" Anne laughs and says, "No."

6.4. Frisbee

6.4.1. (Time [0:10:06]) They begin to play with the Frisbee. A strength of 30 gives 20 and then 40 and then 30. 60% strength gives 50, then 70, then 60. 90 % strength gives 90. 100% gives 100, then 100, and then 100 again. Anne says, "100 just keeps going to itself." Another 100% strength however gives 110, and then 90, and then 120.

6.4.2. Anne says, "It could be a wind or something .... because most of keeps going to 100 then it got down, up a bit; the wind could have stopped so it could go much further. 60% gives 80 then 20% gives 10, then 30 then 30 again then 20.

6.4.3. Anne says, "It's probably controlled to do that or something .... to go on a certain distance at different times." Rebecca: "I'm not too sure really, except that it seems to go to one for a bit then it will change to half way ...." Anne interjects, "That's why I think it's controlled." I ask if they think they can control it. Rebecca says, "No, sort of .... say if you've got something like 30, if you land it in the middle, it will probably land on 40 next." She tries this but it does not quite do what she predicted. I ask for clarification. Rebecca says, "Say, if you do 30, it will probably go into the middle first then to one before and then the one after." I ask if this one is random. Rebecca says, "No ....Same reason again."

6.5. *Roll-A-Penny*

6.5.1. (Time [0:14:18]) They begin to use the roll-a-penny. Strength 80% gives a distance of 80 three times in a row. Anne says, "Keeps going to 80, this one." A fourth go at this strength still gives 80. A strength of 100% gives 100, three times in a row. Anne says, "I think the number it is on it will go to that number." 100% gives 100 and then 90% gives 90. Anne says, "This one might be controlled, sort of going on the number it is meant to be on." 70% strength gives 70. Rebecca agrees with Anne.

6.5.2. I ask if this one is random. Rebecca says, "No, because the strength that you throw it at, it goes to the representative number on the board." Anne is still unable to offer an opinion on randomness.

6.5.3. I ask, "Is the roll-a-penny similar or different from the Frisbee?" Anne says, "It's a bit different, I think. Because that one kept going to different places; this one keeps going to the strength on the number." I ask which is more realistic. They both say that the Frisbee is more realistic. Anne adds, "Because that one *(referring to the roll-a-penny)* is not really realistic because it keeps going to the strengths of the number." I ask what it should do. Anne: "Well, go to different numbers all the time." Rebecca agrees. They both say that they could predict what comes next on the roll-a-penny.

6.6. *Introducing the tools*

6.6.1. (Time [0:17:04]) I use the coin as an example to show them the various tools and resources, which might help to decide how the gadgets are behaving, and how to mend them. I emphasise that, if we think the real device behaves randomly, we want to make the computer's gadget behave randomly, or at least to look as if it behaves randomly.

6.7. *Coin*

6.7.1. (Time [0:21:15]) They begin to use the coin. After a couple of clicks, they use the repeat to do 3 trials. After several more clicks and repeats, they have done 27 trials in all. They open up the results box. Rebecca says, "Lots of tails *(pointing to a sequence of five tails in the last six results)*." I ask, "What do you feel about those results? What do those results tell you?" Anne: "Well, mostly it comes with tails." Rebecca adds, "As we repeated it, tails most of the time." I ask, "What do you feel, Rebecca, about what you were saying before that it would work to a pattern? Do you think there is a pattern?" Anne in fact responds, "Oh yes, I can see a bit of a pattern. Because that's got head, head, head, tail, and it's got .... oh, where was it? I can't think .... oh yes, it's got a head, oh no, that's not right, it's got tail, tail, tail, tail, that time." Rebecca: "I'm not too sure." Anne adds, "I don't think there's much of a pattern really." I ask, "Did you expect there to be a pattern from what you were saying before." Rebecca says, 'Yes, I did." Anne says, "Not really."

6.7.2. They look at the pictogram which shows slightly more tails. The pie chart shows that there are very nearly the same proportion of heads and tails. Anne says, "Oh, it's nearly in the middle. There's a tiny bit more, fraction more." I ask, 'How do you think that compares to what you might expect for a real coin?" Anne: "I think it might be different because I think we probably control it .... actually, I don't know." Rebecca: "I'm not too sure because .... it might be .... the same or it might not." Anne adds, "A real coin sort of might be fair because you can't really estimate what it's going to come out, and you can't estimate on that really." Rebecca agrees. Anne continues, "Because you don't throw it with the same strength all the time like you see on the computer, do you?"

6.7.3. I suggest that they play some more. After 35 trials they look at the results box. Rebecca asks whether there is a limit to how many times you can toss the coin. I say that there is no real limit as the limit is very large, and ask, "If you are trying to decide whether the coin is working properly or not, what do you think a good number of times to throw it would be?" Rebecca says, "About ten." I ask, "Do you think there would be any advantage in doing it more than ten?" Rebecca: "Yes because you get like a clearer answer." I ask how many times would be better than ten. Rebecca: "Twenty." Anne agrees. I ask again if more than 20 would be advantageous. Rebecca says, "Yes." Anne says, "I don't think so."

6.7.4. I point out that we can do lots of tosses quickly on the computer. Starting from fresh, they toss the coin 50 times with a new strength of 50. The pictogram shows more tails, confirmed by the pie chart. I ask how it compares to a real coin. Anne says, "I don't think it would be quite like that .... it would be different numbers, not really even." I point out that this is not even. Anne says, "I don't think with a real coin you'd be throwing it fifty times to see if ...." I suggest that you could do and ask again if that would be the sort of picture that you might get, asking, 'Is it possible that you would get that picture, do you think?" Rebecca: "Probably you could get that result." Anne: "But it might be more heads than tails, you know."

6.7.5. I ask, "What do you think the workings is saying? What do you think the workings is telling the computer to do?" They both suggest that it is saying to get more tails than heads. I ask whether the workings seems to be saying that. Anne: "Yes, definitely." Anne beings to read off the results, making it clear that she is looking at the results rather than the workings. Rebecca: "You could choose if it was going to be a head or a tail." I ask how it chooses. There is a long silence. Eventually, Anne says, "It probably could .... er .... put tails in more or heads in more, something like that." Rebecca: "I'm not to sure, it could just sort of send like a cursor going bleeping from one to the other, and tell it to stop at a random time." I ask, "So you think it is choosing randomly?" Rebecca: "Sort of. Probably."

6.7.6. I suggest that they double-click the command **choose-from** in the workings box. It returns Head. They did this lots of times. I ask if that is choosing randomly. There was little response to this. I ask Rebecca to tell Anne what she thinks *randomly* means. Rebecca: "Not controlled. So, if you do flip a coin, it could land on heads and tails and it wouldn't be a controlled answer. You can't control what's going on. It's random really."

6.7.7. (Time [0:33:02]) I ask if they think this coin is working properly. They think not. Anne: "On the computer, it can choose which one it wants. You can't really choose." Rebecca: "You can't decide yourself whether it's going to be heads or tails." I explain again that I am trying to programme the computer so that it behaves as much as possible like a real coin, so that we can't distinguish the results from a real coin. Rebecca: "I suppose if you used the choose-from, you could probably choose it by clicking on that and telling it to throw, it would probably throw the one that you chose." I confirm that this is what happens.

6.7.8. I ask, "So really, what I am asking you is, 'Do you think I have got it right? Do you think that this program is working so that these results, that we are getting, are indistinguishable from a real coin, that they look the same as a real coin might, or not?'" Anne: "I don't think so .... there are quite a lot of tails coming out, and not so many heads." Rebecca joins in and agrees with this. I clarify, "So you think with a real coin it would be closer. It would be nearer to 50 / 50, sort of thing?" They both say, "Yes."

6.7.9. I ask if there is any advantage in doing more tosses. They believe there is. They change the strength to 50. They do a new 100 trials. They do a pictogram which shows many more tails. Anne says, "I think the tails is more popular .... It seemed like first there more heads were coming up, but now loads more tails are coming up." I ask if they would get the same picture if they did it again. Anne: "Yes." Rebecca: "I'm not sure. It might be 50 / 50; it might be more heads." Anne: "I think it will be more tails." They o a new 100 trials. The pictogram shows more heads. Anne comments, "Oh .... I don't think you can really estimate which one." Rebecca adds, "You can't be too sure really." I ask whether you could be sure with a real coin. Rebecca: "Well, it depends what side you start on, I think ....If you start on tails, it might land on tails again because it might not be a very good flick. If it's heads, and you flick it, and it isn't a very good flick, it will land on heads." I ask if this is what she does in games. She says not, except she has a double-headed coin.

6.7.10. After looking at the pie chart, Anne says, "The first time there were loads of tails, so I thought it was going to be tails again. But probably after a couple of goes, it will probably do tons of heads again." I ask, "What do you think will happen if we do it 200 times?" Anne: "Probably it will be about even, I think." I ask why. Anne says, "Well, because if it is quite a lot more, there might be more chance it will be even really." Rebecca says she is not too sure. They change the strength to 100, and do 200 trials. They do a pictogram which shows the two rows very close to equal in length. This is confirmed by the pie chart. I ask, "Is that just chance that Anne was right, or ...." Rebecca: "Probably." Anne: "I think there was a reason really because .... I don't know really. I just thought that if we did more there would be more chance of it being even." Rebecca adds, "I think that it's actually a fraction over." Anne says, "Yes, but it's fairly even, isn't it?" They briefly discuss whether it is slightly more heads than tails or vice versa. Anne says, "Yes, but it's only a bit of a fraction. I thought it would be even because there'll be more."

6.7.11. I ask, "So if we did it 500 times?" Anne: "It will probably be about even, I think, because that will be more." Rebecca: "Probably even." They decide to leave the strength as 100. I ask if the strength is having any effect. They talk about the different effect on the graphics and I ask if it affects the results. Rebecca: "Yes. If you do 100, it will have more strength, and it will throw a more random number, maybe. Rebecca changes the strength to 10. They do a new 500 trials. They accidentally do 1000 trials. They are now both predicting that the pie chart will be even. Anne says, "The pie chart can be a bit more easy to see if it's even in the middle, but the pict is a bit easier because you can see it more ...."

6.7.12. (Time [0:44:14]) They look at the pie chart which is almost even but shows slightly more heads. Anne says, "I think it's the highest the number, the even more it gets." I clarify, "The higher the number, the more even it gets." Anne: "Yes." I ask Rebecca if she agrees. Rebecca: "Because the other time, when we did less numbers, it was half um even really." I ask, "Do you agree with that — the more times you do it, the more even it's getting?" Rebecca: "Yes, it seems to be." I ask if that would be true of a real coin. They both think it would. They look at the pictogram. Rebecca says, "Slightly more heads than tails." Anne: "It's fairly even though."

6.8. Spinner

6.8.1. (Time [0:45:34]) Before they begin to play with the spinner, Anne says, "Oh, look, it's got to choose from different numbers." Rebecca: "That's definitely more chance of it landing on 1 then." Anne agrees. Anne: "There's more chance of it because that's *(pointing to the 1 sector)* much bigger than these *(pointing to the other four sectors).*" They make the strength 50 and do 50 trials. The pictogram shows most 1's, then 4's with 3's the least. The pie confirms this. Anne says,. "It's a bit like that really *(pointing to the spinner)*. I thought it would be more because 1's more of a chance of getting there because it is much bigger."

6.8.2. I ask if it is random. Rebecca says, "Not really .... Because there is more chance of it landing on 1, than the other numbers." Anne says that was what she was going to say. I say, "It's interesting though because what you said earlier, when I asked you about what random meant, when I asked you tell Anne what random meant, you talked about not being able to control it. But you are adding something different now, because it's not quite the same, is it? You are saying it's not random here because there's more chance of it being 1." Rebecca says, "Yes." Anne says, "I'm still confused."

6.8.3. I ask them what they would have to do to make it fair. They refer to making the 1 sector smaller to make them all the same size. I ask point out that they can only change the workings. They edit the workings to read: **choose-from [1 2 3 4 5]**. They note that the spinner itself has changed. They do 50 trials. The pie chart shows most 5's and 2's with least 1's though there is not a big difference between the sectors. I ask why the 5 is most. Anne says, "Well, the 5 looks a bit bigger than the others. Oh no, it it's not .... well, it seems a bit even but .... much more even but a tiny bit more on 5, I would say."

6.8.4. I ask, "If you're aim was to make that pie chart look more even, what would you do?" Anne says, "I'd make the 5 a bit smaller ...." Rebecca interjects, "I'd make the others a bit bigger." They begin to edit the workings. Anne says, "Why don't you put them all the same number. That would be even then. Like put three on 1, three on 2. That would be fair because that would be even then." In fact they edit the workings to read: **choose-from [1 1 2 2 3 3 4 4 5 5]**.

6.8.5. (Time [0:50:48]) I ask what the workings are saying. Anne: "Well, I think it is going to be more even now. You can't really estimate what it's going to be on. Much more even." They change the strength to 10 and do 50 new trials. The pie chart shows most 1's and least 4's. Anne says, "More 1's again." Rebecca says, "I wonder what would happen if we took one more away." They edit the workings to read: **choose-from [1 2 2 3 3 4 4 5 5]**. I ask Rebecca what she was thinking when she made those changes. Rebecca: "Maybe if 1 was a bit smaller, maybe it would be a bit more even." Anne says, "I think if you had one of each number, it would be even." They do 50 new trials. The pie chart shows most 3's, least 2's with 1's about average. Rebecca says, "Maybe if that *(pointing to the 1 sector on the spinner)* were a tiny bit bigger."

6.8.6. Anne says she wants to try her experiment. So they edit the workings back to read **choose-from [1 2 3 4 5]**. They do new trials. The pie is fairly even but most 1's and least 2's. Anne says, "It's a tiny bit even but there are more 1's." The pictogram confirms this. Rebecca: "Yes, definitely more 1's." Anne: "Yes, but it's fairly even, if you see what I mean." I ask, "So, if you are trying to make this pie chart more even what could you do?" Rebecca: "Maybe throw it more times like we did with the coin." Anne says, "Yes."

6.8.7. They decide 150. I ask whether it would be better to do 100, or 200, than 150. Rebecca: "Well, yes but gradually." The pie chart for 150 trials is more even with slightly more 4's and 3's, and less 2's. I ask what they put that picture down to. Rebecca: "There's a higher number, so the more chance of it being even, I think .... The more times you throw it, the evener it seems to get. And I think that's because there's more chances for a number to come up than if you do it say 50 times." They do a new 200 trials. The pictogram indicates most 2's and least 4's. This is confirmed by the pie chart, though this type of graph gives a more even appearance.

6.8.8. Anne says, "If we do say about a 1000 ...." I first ask about the workings. Rebecca: "Put it on this number this time and maybe that number the next time." I say, "So you think there is a pattern to the results?" Anne: "Yes." Rebecca: "Maybe." I suggest that they look at the results box. Rebecca says, "5 5 seems to appear quite.... double figures keep appearing .... double fives." I ask, "You think there are more double 5's or doubles there than you expect to get?" Rebecca: "Yes." Anne says, "There don't seem to be too many 1's."

6.8.9. Now they change the strength to 100 and they try 1000 trials. I ask, "What do you think the pie chart is going to look like?" Anne: "Sort of even." Rebecca: "Even, yes, because the more times you do it, the more even it seems." The pie chart shows even sectors. Anne: "Yes, that looks even." Rebecca agrees and adds, "It looks just like the spinner does." I ask if they think the spinner is fair. They believe it is.

6.9. Dice

6.9.1. (Time [1:02:14]) They begin to use the dice at top level, and, after just a few clicks, Anne mentions that 6 comes up more. They soon open up the dice and repeat 50 trials. I ask them what the pictogram will look like. Rebecca, looking at the results, says, "Quite a lot of 6's there." Anne: "Maybe a bit uneven.." They look at the pictogram, which shows more 6's and with quite a lot of 2's. The pie chart confirms this. I ask why they are getting more 6's. Anne: "Because of the strength."

6.9.2. They decide to try a new 1000 trials, with a strength of 100 (it had been 50). I ask, "What's the advantage of doing it 1000 times?" Anne: "You get more and you can sort of estimate." I ask what will happen. Anne: "Maybe a bit even." Rebecca: "Maybe it's going to be even again because it seems to go more even the more times you throw it." Anne: "I think it's the more you throw it, the more even it gets." Rebecca: "Yes, that seems to be the case." Anne: "Because that's what happened most of the times, the more you get, the more even you get."

6.9.3. In fact, the pie chart shows many more 6's with the other sectors about equal. Anne says, "I think 6's is popular because there's quite a lot of 6's in the choose-from *(and points to the workings box)* .... There might be a lot more 6's so it's got more chances of getting more 6's on it." The pictogram confirms the image that more 6's are appearing. I ask, "So do you think that's behaving like a real dice should." Rebecca: "Not really."

6.9.4. I suggest that they try to mend it. They begin to delete some of the 6's when Anne says, "Let's have two numbers .... Shall we have two of the same number or just one?" I ask, "Do you think choose from 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6 is any different from choose from 1, 2, 3, 4, 5, 6?" Anne: "Yes .... Because it has double numbers so there's more chance of getting loads more of them, and, if it just has one, you can't get any so there won't be much more ...." Rebecca: "No, I don't think so .... It's roughly the same because it's just the same numbers but just doubled. There's the same amount of chance of getting a 6 as with double numbers." Rebecca has now edited the workings box to read: **choose-from [1 2 3 4 5 6]**. Anne argues: "Well, I think it would be even as there's not doubled numbers." I ask, "What would it be if there were double numbers?" Anne: "Well, it would probably be a bit more than the others." I ask which would be more. Anne: "The doubles."

Rebecca objects, "But they're all doubles." After a pause, Anne says, "It would probably be about even." Rebecca: "It's the same, isn't it." Anne responds, "Oh, I thought you said 1, 1, 1."

6.9.5. They decide to do 1000 trials. I ask them what it will look like. Anne: "Very even." Rebecca: "Yes, roughly even .... I think there's more chance of getting other numbers. Well, a 50 / 50 chance." The pie chart shows even sectors. Rebecca: "Oh look, it's lovely." Anne: "I was right again." I ask, "So, doing it lots of times, like 1000, has made it more even this time, but, before we altered it when the workings were as before, it didn't make it more even. So, what do you think it did do?" After some confused attempts to explain by Anne, Rebecca says, "Because there's more chance of getting a 6. When it stops it might land on a 6." Anne says, "Because the workings were unfair."

6.9.6. I ask, "Let's say we were playing a game, and for some peculiar reason in this game, it would have to be a computer game because we are using the computer dice, we wanted there to be a good chance of getting 1's, and a fairly good chance of getting 2's but a pretty low chance of getting anything else. It's a strange game. How would we make this dice behave like that." Rebecca says, "You have to put more of the numbers on here." She begins to edit the workings until they read **choose-from [1 1 2 2 3 4 5 6]**. They test this out by repeating 1000 trials. I ask what the pie chart will look like. Rebecca: "More 2's, more 1's and less of the others." I ask how the 1's and 2's will compare. They say they will be roughly even. The pie chart comes out as expected.

6.9.7. (Time [1:15:18]) I ask, "If we had done it only 50 times, instead of 1000, do you think we would have got a picture like that?" They both say not. Anne adds, "A bit more uneven." Rebecca says, "There'd be more 1's and 2's." Anne says, "They'd probably be about even." Rebecca says, "Maybe."

6.9.8. I point out that what I had really wanted was for the 1's to have a very good chance and the 2's to have only a fairly good chance. Rebecca immediately edits the workings box to read **choose-from [1 1 1 2 2 3 4 5 6]**.

6.10. Two-Spinners

6.10.1. (Time [1:17:00]) They begin to play with the two-spinners gadget. A strength of 60% gave 2+2. 10% gave 2+1, then 3+3, then 1+1. 10% strength gives 3+1. Rebecca says, "So they're the sums it does." I ask what she is referring to. Rebecca: "The workings." I ask what the workings are saying. Rebecca: "That's the sums it does *(pointing to the data in the workings box)*. It doesn't do different." They carry out some more spins, varying the strength from time to time.

6.10.2. I ask if they can predict what is going to come next. They both say, "No." After 12 goes, they put the strength to 50 and repeat 50 new trials. I ask, "If these were real spinners, made like this so that each section was equal, the different totals that we could get, do you think there is any total which is more likely than any other total?" Before they can answer they have looked at the pictogram and the pie chart, which show most 4's and 3's and less 2's and 6's, but no 5's. Rebecca says, "I'm not sure really. There seem to be lots that seem to come to 3 and 4." I persist by checking that they understand that all totals between 2 and 6 are possible, and then asking, "If these were real spinners, and we can get any total between 2 and 6, do you think there is any total is harder to get than the others; any total that's easier to get than any others?" Anne: "What it was in real life? *(I confirm.)* No." Rebecca: "There's a 50 / 50 chance of getting any total." I clarify, "So you think all the totals are equally easy or hard to get." They both say, "Yes."

6.10.3. They look at the pie chart of their current experiment and comment on the fact that there are more 4's and 3's. Rebecca says, "Maybe if we do it more times, it might be more even. It was with the dice and the spinner." They change the strength to 100 and do 1000 trials. I ask what the pie chart will look like. Rebecca says it will show lots of 3's and 4's. Anne says it might be fairly even between the 3 and 4 but the rest a bit smaller. Rebecca adds, "Because most of the sums seem to come to either 3 or 4." I ask her what she is looking at and Rebecca says, "The workings .... The second one down comes to 3 and then the third one down comes to 3. Then the next two down both come to 4." The pie chart shows most 3's and 4's and least 2's and 6's.

6.10.4. They look at the pictogram and notice that the 5 is missing. I ask why. Rebecca says, "None of the sums come to 5." Anne says, "I think that's because there are no numbers that you can get to 5." Rebecca: "No, you can have 2 and 3." They begin to edit the workings to read **choose-from [1+1 1+2 2+1 2+2 3+1 2+3 3+2 3+3]**. I ask, "Why do you think it is important to have 2 plus 3 as well as 3 plus 2?" Rebecca: "Because the first number is representing the first spinner, so you have to have it both ways." I ask them to check carefully whether they have got them all. As they do this, Rebecca says, "The double numbers you can't really do the other way round." They decide they have got them all (not noticing the missing 1+3).

6.10.5. They repeat a new 1000 trials. I ask what the pie chart will look like. Anne: "Fairly even." I clarify, "So all the totals 2, 3, 4, 5 and 6 all about the same?" Anne: "Should be." The pie chart shows most 5's, 4's and 3's, then 6's and least 2's. Anne says, "Maybe we should put 1 add 1 in again." Rebecca objects, "No, because it's already been done so it would just be the same." Anne argues, "Yes, because some of them have got the same again." Rebecca agrees "Yes, maybe we should actually put some of them in again, because then there's more chance of them coming out more even." I ask Anne what she means when she says that some of them are the same. Anne: "Well, 1 and 2 and 2 and 1 are the same .... they come to the same number." I say, "They come to the same total, but are they the same as far as the spinners are concerned?" Rebecca explains, "No they are not. Because, the second one down, that number *(pointing to the 1 of 1+2 in the workings box)* refers to that spinner *(pointing to the first spinner)*, and that number *(pointing to the 2 of 1+2)* refers to that spinner *(pointing to the second spinner)*. So, say, if that one *(the first spinner)* lands on 1 and that one *(the second spinner)* lands on 2, it would be three. And if that one *(the second spinner)* lands on 1 and that one *(the first spinner)* lands on 2, it would be three as well." Anne says, "Exactly .... I think we should add that one *(pointing to the 1+1)* and that one *(pointing to the 2+2)* again because then we get more of a chance of getting them." I say, "But then you would be putting in 1 plus 1 in twice." Anne responds, "Yes, because 2 doesn't come up as much, does it?" Rebecca agrees, "So maybe if we do that."

6.10.6. They begin to edit the workings. At one point Rebecca is about to add 2+2 as well but Anne points out that that would get 4 then and they already have loads of 4's. Finally, the workings read: **choose-from**

[1+1 1+1 1+2 2+1 2+2 3+1 2+3 3+2 3+3 3+3].

6.10.7. (Time [1:34:44]) They repeat 1000 new trials. The pie chart looks fairly even. I point out, "What we don't know for sure is whether that is how real spinners would behave. I think what you need to try and do is justify why you should have 1 plus 1 in there twice over." Rebecca: "Because everything else has two ways of coming except maybe 2 plus 2." I say, "But in reality, does 1 plus 1 have two different ways of coming?" They both say, "No." I say, "We are trying to make this behave like a real one." Anne says, "On the other chart, the 2 wasn't coming up, so ...." Rebecca interjects, "I think it is more fair because the pie chart looks roughly even and before there were barely any 2 and barely any 4 ..." I say, "I think you have certainly made it more fair. What I am not convinced about is that you have made it more like real spinners would be." Rebecca: "There's more of an even chance of getting it." I say, "I think you've done that, yes, but maybe with real spinners that would not be the case." Anne: "Oh, yes, mmm."

6.10.8. I say, "You see, I am not sure you are being fair by putting 1+1 in twice." Anne says, "We don't want it to be even. We want it to work like a real spinner." I say, "Yes, which may or may not be even." After this heavy guidance from me they edit the workings to read, **choose-from [1+1 1+2 2+1 2+2 3+1 2+3 3+2 3+3]**. I say, "What about 1 plus 3?" Anne: "Oh, yes." They insert 1+3 into the workings." They do 1000 new trials. I ask, "What do you think this pie chart will look like?" Anne: "A bit uneven .... because 1 and 1 has only got once, because that is what a real spinner would be like. And the rest has got like double number and it can make different numbers. " Rebecca: "I think maybe .... 2 won't come up as much and 6." The pie chart shows most 4's and least 2's and 6's, with slightly less 6's than 2's.

6.10.9. I ask why the 4 should be bigger. They look at the workings. Rebecca says, "I've seen it twice .... It's written there three times." I ask how many ways of getting 3. They find two ways. I ask why 4 is coming up most often. Anne: "Because it's got more of the numbers. It's got like three different numbers so it's coming up much more." I ask, "Now, how do you think this compares to doing it two real spinners?" Anne: "Probably it would be about the same because we are trying to work it as a real spinner, and we've got the same sort of numbers." Rebecca: "I agree with Anne." I ask, "So do you think the different totals, 2, 3, 4, 5, 6 are all just as easy to get or all just as hard to get. Or is there one that's easy to get?" They both say 4 is easier to get. And I ask if there is one that is hard. They both say, "6". I ask, "Just the 6?" They then say, "2". Rebecca adds, "But 6 seems harder because it is smaller ...." Anne interjects, "Yes, much smaller." Rebecca continues, "Less of 6."

6.10.10. I suggest that they repeat the experiment to see if they get the same picture again. I summarise that they think the 6 is harder because there are less 6's than 2's. Rebecca responds, "There seems to be less ways of getting 6 and 2 than there are of 4, 5, and ..." Anne adds, "It is even really because you can get different ways of getting 4, there's three different ways, it's really fair really." The new pie chart is essentially the same but the 6's beat the 2's this time. I ask how the 2 and the 6 compare. Rebecca: "The 2 is smaller this time than last time. Because last time 6 was smaller." I ask, "So do you think 2 is easier to get, just the same, or harder to get than the 6." Rebecca: "Just the same, I think, because last time 6 was slightly bigger than 2 and this time.... the other way round, 2 was bigger then 6 and time 6 is bigger than 2." Anne: "Yes, I agree."

6.10.11. I ask what the workings suggest about 2 and 6. Rebecca: "There's only one way of getting them, and there's two for 3 and 5, and three ways of getting 4." I ask, "So, if we were going to play a game, in which we had two spinners like this, and you are going to bet a pound and I am going to bet a pound. And you're going to bet that a total of 4 comes up, and I'm going to bet that a total of 6 comes up. Would you take that bet?" Anne: "No .... well, sort of .... I wouldn't bet as much as a pound." I ask how much would she bet. Anne: "20p." Rebecca: "2p." I ask, "If we were to bet 2p the other way round that I'm betting on a 4 and you're betting on a 6, would you take that bet?" Rebecca: "No." Anne: "Definitely not."

6.11. Two-Dice

6.11.1. (Time [1:46:43]) They begin to play with the two-dice gadget. They use a strength of 20 and this gives 2+2. 100% gives 1+1. I ask, "Without shaking the dice any more, do you think that these two dice are behaving like two dice should?" Anne: "I don't think it's got all the numbers in there *(pointing to the workings)* .... I'm not sure .... I thought something was missing, I'm not sure .... I thought there would be more than that" Rebecca: "It's got something's missing because it isn't a very long list, I think." Rebecca adds, "There's lots of different ways .... There's probably more ways of making 6 .... " Anne points out, "Ah, 5 add 1 is missing to make 6." They insert 5+1 below 1+1.

6.11.2. They roll the dice and in fact they get 5+1. Rebecca says, "That was lucky." They roll again and get 5+1 again! I check that they recognise the range of totals as going from 2 to 12, and ask, "If we were shaking two real dice, do you think all the totals you could get are just as easy, just as hard, or do you think some totals are easier than others, harder than others?" Rebecca begins, "Well, on the computer ...." I interrupt, "With two real dice." Rebecca: "50 / 50 chance of getting them." Anne: "Yes." I clarify, "So you think they are all about the same chance?" They both say, "Yes."

6.11.3. Another roll gives 5+5. Rebecca: "You can get 9 can't you .... I can't see 5 add 4, or 4 add 5 anywhere." They insert 5+4 and 4+5 beneath 1+1. Anne spots 2 add 3 is also missing. Anne points out that they need 3 add 2 as well. They insert 3+2 beneath 6+6. The workings now read: **choose-from [1+1 5+4 4+5 2+3 5+1 1+2 2+1 2+2 3+3 4+4 5+5 6+6 3+2]**.

6.11.4. Rebecca suggests that they repeat it a number of times to get a clue which ones are missing. They repeat 1000 new trials. I ask, "What do you think it *(referring to the pie chart)* should look like if it were two real dice?" They both say that it will be fairly even. The pie chart is uneven and shows most 6's and least 2's.

6.11.5. (Time [1:55:02]) They look at the pictogram. This shows that there are no 7's and 11's. Anne says, "It hasn't got 7. We need something to get 7 and 1. Oh no, you can't get 1, can you? .... 7 and 11." The insert 6+5 and 5+6 at the bottom of the workings box. Then they append 5+2, 2+5, 3+4, 4+3, 6+1 and 1+6. I suggest that they check the other total systematically. They work through totals 2 through to 12. Eventually they have all 36 combinations.

6.11.6. (Time [2:06:48]) They repeat 1000 new trials. I ask what they think the pie chart will look like. Anne: "Fairly even? ....Some of the numbers might not be because there's not as much as the other number." Rebecca: "Maybe roughly even because now that we have got all the sums. I'm not too sure at the moment." Anne adds, "I think some will be a bit less because they haven't got as much as the others .... because some of the numbers will not be the same, will be less, because we didn't find enough sums for them .... like 1 add 1." I ask, "Can you give me an example of one that had a lot of different ways of getting it." Anne: "7."

6.11.7. The pie chart shows most 7's and least 2's and 12's. Rebecca's first reaction is that they have not got them all. I reassure her that we have. Rebecca says, "Ah, I bet there are various ways of making a number. There can be more ways of making one number than there can be of another." I ask them which numbers have not many ways of being made. Anne: "12 and 3." Rebecca: "12 and 2." They click on the pictogram button. I ask, "Which one is going to be the longest?" Anne: "The 7." Rebecca: "I agree with Anne. Probably. I'm not too sure." A bug prevents the pictogram from appearing.

6.11.8. We return to the pie chart and I ask why it looks like that. Anne: "Because some of the sums we put as more. Like 2, we could only get about one of them so that's why it came out a bit like that." I ask, "So when you home, and you are playing dice with your brother, Rebecca, and you've got two dice, and you say to him, 'I bet you 10p what the total's going to be.' You go for 7. What are you going to tell him to go for?" Rebecca: "2 .... no, sorry 12 .... 12 is smaller than 2." I ask, "So, do you now think that all the totals on two dice are just as easy or just as hard as each other? Or do you think some totals are easier than others?" They both say that some totals are easier to get.

6.11.9. Rebecca says, "It doesn't seem random now, does it?" I ask, "If you shake two dice at home and you are looking at the totals that you get, do you think they are random?" Rebecca: "Not really, well, not really random because they are not 50 / 50." I say, "So you can't control what's happening, but they're still not random because they're not all the same chance. That's the way you are thinking of it?" Rebecca: "Yes, it's not 50 / 50 because you don't get the same number."

6.11.10. (Time [2:16:20])

NOTE : THE OTHER CASE ACCOUNTS CAN BE FOUND ON THE WORLD WIDE WEB AT ADDRESS: http://www.warwick.ac.uk/wie/staff/DP.htm

A3.5. Case Analysis of Anne's and Rebecca's clinical interview

The profile draw on the pre-interviews. The remainder of the case analysis draws on the case account for Anne and Rebecca. The coded references refer to the corresponding paragraphs in that case account.

PROFILE OF ANNE

Age: 10.6 years

Anne says that she has not heard of the word 'random'. Nevertheless, she recognises that no number on a normal is dice is more likely to appear than any other though she can offer no explanation for that. She also says that she can not control the outcome of the dice, other than by rolling it gently.

In order to test the fairness of a dice, Anne suggests rolling the dice a few times "because it would probably come up the same number every time". She goes on to suggest that about three throws would be enough. Anne underestimates the number of throws needed even if the unfairness takes the form of the same number appearing constantly. She makes no mention of other, perhaps more subtle, forms of unfairness.

On the uniform spinner, Anne suggested that no number was harder to get than the others "because you can't really know what number's going to come up". She does though suggest that 5 is easier to get. This is on the basis that in a couple of trial throws the 5 came up both times.

On two dice, Anne says that there is no total which is harder or easier to get than the others. she explains, "Because you can't estimate what number you'll get because they're all fair, both the numbers are fair."

Anne regards the totals as fair because each dice is fair, and so concludes that, if the totals are fair, then they must be equally easy or hard to obtain.

Anne also sees doubles as harder to obtain than a total of 3. She explains, "Because if you throw two dices it normally comes out a different number, it doesn't normally come out a double number." She explains that a double is harder to obtain than a total of 7 and a total of 5 for the same reason

Anne treats the compound event, a specific total, as if it were a special case of the event *two different numbers.* For this event, her responses are correct. She is not alerted to other interpretations by the fact that I keep repeating the question using different totals. Even though I was careful in the language of my questions to be quite explicit that we were talking about any double and about a specific total, it is possible that Anne misinterprets my questions. It seems to me that it is more likely that she is unaware of the possibility of these other interpretations.

PROFILE OF REBECCA

Age: 11.0 years

Rebecca refers to a dice as random "cos you can't control which number it goes on". She also says that no numbers on the dice are easier or harder to get because it is random. In her response to testing a possibly unfair dice, Rebecca suggests "you could roll it a few times, maybe ten or twenty times and if you get the same number, or one number more times than others".

Rebecca says that the uniform spinner is random "cos there's the same amount of space for each number whereas the non-uniform spinner is not random "cos there's more apace for 1 and less space for 4".

It is clear that Rebecca sees randomness in terms of lack of control but also in terms of equiprobability or fairness. She recognises that unfairness may occur through one number occurring more times but thinks that she can test for this in 20 or less throws. This underestimation may suggests that she does not appreciate the large number behaviour of dice.

She uses this same type of argument for two dice. She claims that there is no total which is harder to get and no total which is easier to get when rolling two dice. She explains, "Cos it's random, you can't control which number it lands on." The unsteerability intuition suggests to her that the result is random and this in turn means that it is fair, the totals are equiprobable.

In comparing the chances of throwing a double to a specific total, she gives the same response whether the total is 3, 5 or 7. In each case, she argues that because it is random, you can't control which number it lands on and so the double is just as easy or hard as the specific total.

Rebecca also sees the distance travelled by the Frisbee as random. She recognises that some degree of control is possible, arguing "it depends how far you throw it. If you throw it lightly it won't go as far as if you throw it hard". She also realises that this control is limited because of other factors like the wind. The Frisbee can be controlled "to an extent". Therefore she argues that the distance is random "because you can't really control what happens."

In some cases, Rebecca refers to a situation as being "in between" random and not random. For example, in the roll-a-penny case, Rebecca says that the distance travelled by the coin is no really random, explaining, "well, it might go the sae distance but it might not, so it's in between".

Similarly, when she discusses pulling a ball from a bag containing four red balls and one yellow ball, she says it is sort of random "because there's more reds than yellows, so there's not much chance of you pulling out a yellow, there's more chance or you pulling out a red".

Her view that randomness is identifiable primarily by unsteerability suggests to her that, as with the Frisbee, the coin may be random. In competition with this idea is that she also recognises that the coin often goes the same distance (or nearly the same), so there is quite a strong sense of control. This is different perhaps from the Frisbee where the wind factor in particular diminishes the sense of control and allows her to think of this phenomenon as random. The same type of argument explains why she sees the case of pulling out a coloured ball as in between random and non-random. She expects to pick out a red ball and so feels in control but she also realises that she can not be sure. This in between state resolves her difficulty of allowing a situation which is clearly not fair, in the sense of not equiprobable, to be random.

Rebecca emphasises how important steerability is as she tries to discriminate randomness when she summaries her view of randomness in these words, "Well, you can't control something that is random, so it could be anything, anything could happen, so if you're throwing a dice it could be any number." We see how it is unsteerability in her view that leads to unpredictability and this is what she means by random.

ANNE AND REBECCA'S INTERACTIONS WITH THE CHANCE-MAKER MICROWORLD

Local Meanings in the Chance-Maker Microworld

Predictability, regularity and steerability

We have seen in the profiles that Anne does not know what the word random itself means but recognises unpredictability as an important part of the functioning of a spinner or a dice. Rebecca has a strong sense of what random means and it is closely related to unsteerability and therefore unpredictability. Because of this close relationship between steerability and predictability in Rebecca's view of randomness, I have chosen to deal with these local meanings at the same time. The separation of these three aspects would have misrepresented what we can discover about Rebecca's local meanings for the behaviour of stochastic phenomena.

She also sees fairness as an important aspect of randomness so that each outcome should be equally likely. Rebecca has a sense of some phenomena being in between random and non-random and this she sees for situations which are not steerable but which are not uniform.

In their initial interactions with the coin, the two girls generate a tail and then four heads. Anne responds to this by suggesting (6.2.1), "It's a bit unfair because it keeps going to the queen *(i.e. heads)*." Rebecca argues (6.2.2) that the gadget is not really random because "it's probably been programmed to do it, in a loop .... it's programmed to do heads, then maybe heads again and then tails." I clarify, "In some sort of pattern?" Rebecca agrees. I ask if she was able to pick up some sort of pattern. Rebecca says, "Not really."

There is evidence here that both Anne and Rebecca see irregularity as an important aspect of the behaviour of real coins, although Anne is unable to associate this with the word *random*. Rebecca believes that there must be some sort of pattern on a computer, even though she can not actually pick it up.

Both Anne and Rebecca accept that they can not predict the outcome (6.3.2). Anne wonders whether there is a sense in which she can control the spinner, "Sort of .... It normally lands on 1. The slower you go it seems to land on 1 as well." When I press her however, she says that maybe this does not mean that you can control the outcome of the spinner. Anne is trying to express the fact that 1 comes out more often, even when the spinner is spun only lightly (unlike a real spinner). Rebecca thinks that she can not control the result. She argues, "I don't think you can really control it, because if you do it slowly, you never know; it could come on a 2, if it starts off on a different number."

Nevertheless, Rebecca says that the coin is not random. She says, "It's probably programmed." We gain here an interesting insight into Rebecca's view of control. We know from her profile that this aspect of behaviour has been important to her in judging randomness. Here, we see Rebecca rejecting steerability as the discriminator in favour of some less tangible notion of computer control. The computer is in control even though she is not and so the spinner is not random.

Their initial interactions with the Frisbee gadget follow a similar pattern. Anne is quick to spot patterns suggesting, after three successive results of 100, that "100 just keeps going to itself" (6.4.1). By this, she means that each time she uses a strength of 100% to throw the Frisbee the Frisbee also travels a distance of 100. Although this is soon countered by results other than 100 (6.4.2), it shows that Anne is tuning into the possibility of regularity as a discriminatory attribute of the gadget's behaviour.

Anne picks up on Rebecca's notion of computer control (6.4.3) when she comments, "It's probably controlled to do that or something .... to go on a certain distance at different times." Rebecca responds, "I'm not too sure really, except that it seems to go to one for a bit then it will change to half way ...." Anne interjects, "That's why I think it's controlled." I ask if they think they can control it. Rebecca says, "No, sort of .... say if you've got something like 30, if you land it in the middle, it will probably land on 40 next .... Say, if you do 30, it will probably go into the middle first then to one before and then the one after." I ask if this gadget is random. Rebecca says, "No ....Same reason again."

Rebecca recognises that there is some element here of steerability. She can influence the result but it is not entirely predictable. As with the previous gadgets though, the importance of steerability is undermined by the notion of computer control. The Frisbee is not, in Rebecca's view, random.

They are then introduced to the tools in the coin gadget (6.6.1) and this opens up new possibilities for us to understand Anne and Rebecca's local meanings for stochastic behaviour, and in particular Rebecca's notion of control.

After 27 clicks with the coin gadget, they look closely at the results (6.7.1). They comment about there being more tails and than I ask, "What do you feel, Rebecca, about what you were saying before that it would work to a pattern? Do you think there is a pattern?" Anne in fact responds, "Oh yes, I can see a bit of a pattern. Because that's got head, head, head, tail, and it's got .... oh, where was it? I can't think .... oh yes, it's got a head, oh no, that's not right, it's got tail, tail, tail, tail, that time." Rebecca says, "I'm not too sure." Anne adds, "I don't think there's much of a pattern really." I ask, "Did you expect there to be a pattern from what you were saying before." Rebecca says, 'Yes, I did." Anne says, "Not really."

Anne looks for a sequence in the results but finds that the pattern is not supported in the longer term. Rebecca is surprised that she can not find a pattern in the sequence of results. It turns out that this experience begins to undermine her view about computer control.

When the pictogram shows slightly more tails, I ask (6.7.2), "How do you think that compares to what you might expect for a real coin?" Anne replies, "I think it might be different because I think we probably control it .... actually, I don't know." Rebecca says, "I'm not too sure because .... it might be .... the same or it might not." Anne adds, "A real coin sort of might be fair because you can't really estimate what it's going to come out, and you can't estimate on that really." Rebecca agrees. Anne continues, "Because you don't throw it with the same strength all the time like you see on the computer, do you?"

Anne is satisfied that the coin gadget simulates a real coin from the point of view of unpredictability but is less convinced that it is realistic in matters of control.

After trying out some more experiments involving 35 and then 50 trials (6.7.3 & 6.7.4), I ask them what the workings are telling the computer to do (6.7.5). They both suggest that it is saying to get more tails than heads. I ask whether the workings seems to be saying that. Anne: "Yes, definitely." Anne beings to read off the results, making it clear that she is looking at the results rather than the workings. Rebecca says, "You could choose if it was going to be a head or a tail." I ask how it chooses. There is a long silence. Eventually, Anne says, "It probably could .... er .... put tails in more or heads in more, something like that." Rebecca offers, "I'm not to sure, it could just sort of send like a cursor going bleeping from one to the other, and tell it to stop at a random time." I ask, "So you think it is choosing randomly?" Rebecca says, "Sort of. Probably."

For the first time, Rebecca admits to the possibility that the behaviour of the coin gadget may be random. At this point Rebecca and Anne execute several times the **choose-from** command directly (6.7.6). I wonder if this experience will influence their view, which is now a joint view, of the notion of computer control. They both continue to argue that the computer gadget is not working properly 6.7.7). Anne says, "On the computer, it can choose which one it wants. You can't really choose." Rebecca agrees, "You can't decide yourself whether it's going to be heads or tails."

I explain again that I am trying to programme the computer so that it behaves as much as possible like a real coin, so that we can't distinguish the results from a real coin. This is a crucial story for me to insist upon. The girls' attempts to make sense of the coin gadget's behaviour has been important and has acted as a window on their local meanings for stochastic behaviour. Their conclusions regarding the stochastic nature of the gadgets have been less important than what this process tells us about their thinking. Nevertheless, it is important, if the session is to continue being useful that they see a purpose to the mending of the gadgets and this purpose is credible.

Rebecca responds (6.7.7) to my re-presentation of the purpose of this activity by saying, "I suppose if you used the choose-from, you could probably choose it by clicking on that and telling it to throw, it would probably throw the one that you chose." I confirm that this is what happens. I persist (6.7.8), "So really, what I am asking you is, 'Do you think I have got it right? Do you think that this program is working so that these results, that we are getting, are indistinguishable from a real coin, that they look the same as a real coin might, or not?'" Anne replies, "I don't think so .... there are quite a lot of tails coming out, and not so many heads." Rebecca joins in and agrees with this. I clarify, "So you think with a real coin it would be closer. It would be nearer to 50 / 50, sort of thing?" They both say, "Yes." The attention has now switched from matters of computer control to the numbers of heads and tails, which will be more properly discussed under the section on global meanings.

This section has shown how Rebecca and Anne have strong local meanings for the behaviour of stochastic phenomena. These include unpredictability and irregularity in previous results. These intuitions are apparent in the way that they try to search for patterns in previous results. The issue of control is even stronger for Rebecca (and Anne takes on Rebecca's notion of control). Steerability determines predictability; unsteerability allows unpredictability. On the computer, she encounters situations where the unsteerability does lead to unpredictability but believes that the computer must be in control and is surprised when this does not lead to sequences in the results. Rebecca invents a way of thinking about how the computer chooses the result, a method which allows the choice to be random, and at least temporarily resolves her dilemma.

We will see in the next section how Anne and Rebecca will return to matters of control from time to time to explain unexpected outcomes in the long term behaviour of the gadgets, and that it is through a growing appreciation of the limitations of these intuitions that they begin to forge meanings for the long term behaviour of stochastic phenomena.

Global Meanings in the Chance-Maker Microworld

We see in the profiles that there is tentative evidence that Anne expects to be able to identify non-stochastic behaviour in everyday artefacts like coins and dice in just a few trials. We see how Anne expects an unfair dice to show up through the repetition of the same number. This notion of irregularity may be a useful discriminator of short term stochastic behaviour. She does not offer ideas which might suggest well formed global meanings for behaviour of stochastic phenomena.

Rebecca also seems to underestimate how many trials may be required to identify an unfair dice.. Rebecca mentions also the possibility that unfairness might manifest itself as a particular number which simply occurs more often. Perhaps this is why she suggests as many as 20 trials may be necessary. There may here be some very tentative evidence that Rebecca has some global meanings though there is no evidence to suggest that they are well developed.

Simple events

As they interact with the coin gadget, they decide to repeat 35 trials (6.7.3). Rebecca asks whether there is a limit to how many times you can toss the coin. I say that there is no real limit as the limit is very large, and ask, "If you are trying to decide whether the coin is working properly or not, what do you think a good number of times to throw it would be?" Rebecca says, "About ten." I ask, "Do you think there would be any advantage in doing it more than ten?" Rebecca: "Yes because you get like a clearer answer." I ask how many times would be better than ten. Rebecca: "Twenty." Anne agrees. I ask again if more than 20 would be advantageous. Rebecca says, "Yes." Anne says, "I don't think so."

This incident supports the impression gained in the interviews that Rebecca has an intuition that aggregation may be beneficial, though this intuition is still constrained by the experiences of using small numbers. Anne is perhaps less inclined to intuit the power of large numbers in the behaviour of coins.

Anne and Rebecca then carry out 50 trials (6.7.4) with the coin gadget and find more tails than heads. There is some discussion about whether a real coin would generate such results. Rebecca says, "Probably you could get that result." Anne comments, "But it might be more heads than tails, you know." This is followed by some discussion about the workings and whether the coin is random (6.7.5 to 6.7.8).

Anne's and Rebecca's subsequent interactions with the coin, spinner and dice gadgets show very clearly how these initial ideas are re-shaped through the influence of the computer-based resources and my interventions, and how the domain of abstraction is gradually extended.

Summary of paragraphs 6.7.8- 6.9.8

There is some dissatisfaction that so many tails are being produced, and so I ask again (6.7.8) if there is advantage in doing more tosses. This time they both agree that there is (6.7.9).

This time they repeat 100 new trials of the coin gadget. The pictogram shows more tails. Anne says, "I think the tails is more popular .... It seemed like first there more heads were coming up, but now loads more tails are coming up." I ask if they would get the same picture if they did it again. Anne says, "Yes." Rebecca hesitates, "I'm not sure. It might be 50 / 50; it might be more heads." Anne confirms, "I think it will be more tails."

Anne has decided, on the basis that the 100 trials supported the outcome of more tails witnessed after 50 trials, that more tails is a tendency of the gadget and will be repeated in another experiment. Rebecca is less inclined to trust in the results so far, and believes it may still come out about equal. When they repeat a new 100 trials, there are more heads. This is sufficient evidence to persuade Anne that she was wrong, "Oh .... I don't think you can really estimate which one." Rebecca adds, "You can't be too sure really." I ask whether you could be sure with a real coin. Rebecca says, "Well, it depends what side you start on, I think ....If you start on tails, it might land on tails again because it might not be a very good flick. If it's heads, and you flick it, and it isn't a very good flick, it will land on heads." Anne says (6.7.10), "The first time there were loads of tails, so I thought it was going to be tails again. But probably after a couple of goes, it will probably do tons of heads again."

We see how the facility of being able to repeat experiments has enabled Anne to test out her conjecture that tails were appearing more often. She accepts that she can not predict whether heads or tails will appear more often but now conjectures that there is some type of oscillating pattern. Rebecca is not prepared to extend the domain of her abstraction that you can not be sure since in the everyday world the outcome is influenced, she says, by how you flick the coin.

I probe into what they think will happen if 200 trials are repeated. Anne replies, "Probably it will be about even, I think .... because, if it is quite a lot more, there might be more chance it will be even really." Rebecca says she is not too sure. After 200 trials, the pictogram shows the two rows very close to equal in length. Rebecca is unconvinced, suspecting that the evenness may have been just coincidence (6.7.10), whereas Anne thinks there was a reason for this pattern but is unsure as to what that reason might be.

Anne, who appeared to have little intuition for large numbers at the outset has, through a process of rejecting various other explanations for the inconsistent behaviour at large small numbers has constructed an abstraction that larger numbers will generate even results. Rebecca, who seemed to have some initial intuitions about large numbers is not yet convinced of Anne's ideas though she does agree that it will perhaps be even when we carry out 500 trials (6.7.11)

They accidentally do 1000 trials. They are now both predicting that the pie chart will be even. They look at the pie chart (6.7.12) which is indeed almost even. Anne says, "I think it's the highest the number, the even more it gets." I clarify, "The higher the number, the more even it gets." Anne says, "Yes." Rebecca adds, "Because the other time, when we did less numbers, it was half even really." I ask, "Do you agree with that — the more times you do it, the more even it's getting?" Rebecca replies, "Yes, it seems to be." I ask if that would be true of a real coin. They both think it would.

The evidence has now allowed them to construct the abstraction that the higher number of trials, the more even the pie chart becomes. Rebecca is now able to explain the earlier results where there was less consistency by thinking of them as 'half-even'. They are so confident of this abstraction that they are prepared to extend the domain to real coins in the everyday world.

After some initial work with the spinner (6.8.1 to 6.8.2), they edit the workings to make the spinner fair (6.8.3). The workings now read: **choose-from [1 2 3 4 5]**. They repeat 50 trials and find that the 5's and 2's come out most often with 1's the least.

I ask (6.8.4), "If you're aim was to make that pie chart look more even, what would you do?" Anne says, "I'd make the 5 a bit smaller ...." Rebecca interjects, "I'd make the others a bit bigger." They begin to edit the workings. Anne says, "Why don't you put them all the same number. That would be even then. Like put three on 1, three on 2. That would be fair because that would be even then." In fact they edit the workings to read: **choose-from [1 1 2 2 3 3 4 4 5 5]**.

It is interesting that Anne and Rebecca do not respond to my question by suggesting that they repeat a higher number of trials. One might have expected this response based on the intuitions developed with the coin. They were prepared to extend the abstraction that more trials will generate a more even pie chart to the everyday world but not to the spinner gadget. Instead they look towards changes in the workings causing the effect desired.

It is unclear to me why they think that these workings are different from the previous version, so I ask what the workings are saying. Anne responds, "Well, I think it is going to be more even now. You can't really estimate what it's going to be on. Much more even." After 50 new trials, the pie chart shows most 1's and least 4's. Anne says, "More 1's again." Rebecca says, "I wonder what would happen if we took one more away." They edit the workings to read: **choose-from [1 2 2 3 3 4 4 5 5]**. I ask Rebecca what she was thinking when she made those changes. Rebecca replies, "Maybe if 1 was a bit smaller, maybe it would be a bit more even." Anne says, "I think if you had one of each number, it would be even."

Rebecca is responding in a quasi-deterministic fashion, believing that changes in the workings will have direct consequence in the results, when a large small number of trials is used. Anne wishes to try one of each number, though I am not sure that she realises this would return her to the earlier version of the workings.

They repeat 50 new trials. The pie chart shows most 3's and least 2's. Rebecca says, "Maybe if that *(pointing to the 1 sector on the spinner)* were a tiny bit bigger." Rebecca is still calling upon her deterministic intuitions.

Anne says she wants to try her experiment (6.8.6). They edit the workings back to read **choose-from [1 2 3 4 5]**. They do 50 new trials. The pie is fairly even but most 1's and least 2's. Anne says, "It's a tiny bit even but there are more 1's." Rebecca comments "Yes, definitely more 1's." Anne: "Yes, but it's fairly even, if you see what I mean." I ask, "So, if you are trying to make this pie chart more even what could you do?" Rebecca replies, "Maybe throw it more times like we did with the coin." Anne says, "Yes."

Rebecca throws away her previous quasi-deterministic strategy in favour of the abstraction drawn from the coin that more throws leads to more even pie charts. It is interesting to note that the connection between the coin and the spinner was far from obvious for Rebecca. She needed considerable experience with the spinner, forming and rejecting conjectures based on intuitions drawn from her deterministic view of the world, before she was prepared to consider her recently acquired coin-based abstraction.

They decide to try 150 trials (6.8.7). The pie chart for 150 trials is more even with slightly more 4's and 3's, and less 2's. I ask how they explain this picture. Rebecca replies, "There's a higher number, so the more chance of it being even, I think .... The more times you throw it, the evener it seems to get. And I think that's because there's more chances for a number to come up than if you do it say 50 times." They do a new 200 trials. The pictogram indicates most 2's and least 4's.

Anne says (6.8.8), "If we do say about a 1000 ...." I ask, "What do you think the pie chart is going to look like?" Anne replies, "Sort of even." Rebecca: "Even, yes, because the more times you do it, the more even it seems." They repeat 1000 trials (6.8.9). The pie chart shows even sectors. Anne comments, "Yes, that looks even." Rebecca agrees and adds, "It looks just like the spinner does." I ask if they think the spinner is fair. They believe it is.

Anne and Rebecca then begin to use the ice gadget (6.9.1). During the familiarisation process, they comment on how there appears to more 6's, and this is confirmed by looking at the results after 50 trials. They decide now to repeat 1000 trials (6.9.2).

I ask, "What's the advantage of doing it 1000 times?" Anne replies, "You get more and you can sort of estimate." I ask what will happen. Anne says, "Maybe a bit even." Rebecca adds, "Maybe it's going to be even again because it seems to go more even the more times you throw it." Anne agrees, "I think it's the more you throw it, the more even it gets." Rebecca adds, "Yes, that seems to be the case." Anne points out, "Because that's what happened most of the times, the more you get, the more even you get."

Anne and Rebecca have not discriminated the conditions under which their abstraction applies. They over-generalise, not yet appreciating that the evenness of the pie chart is connected with the evenness of the workings box. In this situation, the workings box, to which Anne and Rebecca have so far ignored, contains more 6's.

The pie chart shows many more 6's with the other sectors about equal (6.9.3). Anne says, "I think 6's is popular because there's quite a lot of 6's in the choose-from *(and points to the workings box)* .... There might be a lot more 6's so it's got more chances of getting more 6's on it." The pictogram confirms the image that more 6's are appearing. I ask, "So do you think that's behaving like a real dice should." Rebecca replies, "Not really."

Anne connects the appearance of the pie chart with the data in the workings box. After some discussion about whether to use **choose-from [1 1 2 2 3 3 4 4 5 5 6 6]** or **choose-from [1 2 3 4 5 6]**, they opt for the latter (6.9.4).

They decide to do 1000 trials (6.9.5). I ask them what it will look like. Anne: "Very even." Rebecca: "Yes, roughly even .... I think there's more chance of getting other numbers. Well, a 50 / 50 chance." The pie chart shows even sectors. Rebecca comments, "Oh look, it's lovely." Anne congratulates herself, "I was right again."

I am still interested in whether they appreciate why the pie chart was not even before. I ask, "So, doing it lots of times, like 1000, has made it more even this time, but, before we altered it when the workings were as before, it didn't make it more even. So, what do you think it did do?" After some confused attempts to explain by Anne, Rebecca says, "Because there's more chance of getting a 6. When it stops it might land on a 6." Anne says, "Because the workings were unfair."

I ask (6.9.6), "Let's say we were playing a game, and for some peculiar reason in this game, it would have to be a computer game because we are using the computer dice, we wanted there to be a good chance of getting 1's, and a fairly good chance of getting 2's but a pretty low chance of getting anything else. It's a strange game. How would we make this dice behave like that?" Rebecca edits the workings to read **choose-from [1 1 2 2 3 4 5 6]**. They test this out by repeating 1000 trials. I ask what the pie chart will look like. Rebecca says, "More 2's, more 1's and less of the others." I ask how the 1's and 2's will compare. They say they will be roughly even. The pie chart comes out as expected.

I ask (6.9.7), "If we had done it only 50 times, instead of 1000, do you think we would have got a picture like that?" They both say not. Anne adds, "A bit more uneven." Rebecca says, "There'd be more 1's and 2's." Anne says, "They'd probably be about even." Rebecca says, "Maybe."

I point out (6.9.8) that what I had really wanted was for the 1's to have a very good chance and the 2's to have only a fairly good chance. Rebecca immediately edits the workings box to read **choose-from [1 1 1 2 2 3 4 5 6]**.

Anne and Rebecca now discriminate between cases where the workings are fair, which produce even pie charts when the number of trials is high, and those where the workings are unfair, which will generate uneven pie charts. They discriminate between large numbers and large small numbers, the latter generating less evenness when the workings are fair.

We can track the process by which the two girls construct new global meanings for the behaviour of the coin, spinner and dice gadgets.

Para	Conjectures and Situated Abstractions	Critical Interventions
6.7.8 to 6.7.9	The coin gadget generates more tails (C)	Is there advantage in doing more tosses?
6.7.9	*Activity : A & R carry out 100 trials and the pictogram shows more tails.*	
6.7.9	A: Tails are more popular.	Would you get the same picture again?
6.7.9	*Activity : A & R do a new experiment of 100 trials. This time heads appears more often.*	
6.7.9 to 6.7.10	A: You can't really estimate (SA) R: You can't be sure (SA) R: With a real coin, it depends on how you flick it.	What if we do it 200 times?
6.7.10	*Activity: A & R repeat 200 new trials. The pictogram shows roughly equal rows.*	
6.7.10	R: It may be just chance that it came out even (C) A: I think there was a reason (C)	What if we did it 500 times?
6.7.11	*Activity: A & R repeat another 1000 trials (intending to do 500). The pie chart is even.*	
6.7.12	A & R: The higher the number of trials, the more even the pie chart gets (for the coin gadget and real coins) (SA) R: With less trials, the pie chart is only half even (SA)	
6.8.1 to 6.8.2	*Activity: A & R familiarise themselves with the spinner and find that there are a lot of 1's.*	
6.8.3	The spinner generates too many 1's (SA)	What would you do to make it fair?

6.8.3 to 6.8.5	*Activity: A & R edit the workings to read:* ***choose-from [1 2 3 4 5]****. The pie chart is uneven. They edit the workings to include two of each number. After 50 trials, the pie chart is still uneven.*	
6.8.5	R: Perhaps we should take one more of the 1's away (C)	
6.8.5 to 6.8.6	*Activity: They edit the workings by deleting one of the 1's and do 50 new trials. The pie chart is still uneven.. Anne edits the workings back to* ***choose-from [1 2 3 4 5]****. and they repeat 50 new trials.. The pie chart is fairly even but not entirely convincing.*	
6.8.6	R: Maybe we should throw it more times like the coin (C)	
6.8.7	*Activity: They repeat 150 new trials.. The pie chart is more even.*	
6.8.7	R: The higher the number of trials, the more even the pie chart gets, I think(C)	
6.8.7 to 6.8.9	*Activity: They repeat 200 trials, and the pie chart is a little uneven. They repeat 1000 new trials and the pie chart shows even sectors.*	
6.8.9	R: The more times you spin it, the more even the pie chart gets (SA)	
6.9.1 .	*Activity: A & R begin to use the dice gadget.. There appear to be too many 6's.*	
6.9.2	A & R: If we do it a lot of times, the pie chart will be even (C)	
6.9.2	*Activity: A & R repeat 100 trials and find that the 6's are much more common.*	
6.9.3 to 6.9.4	A: There are more 6's in the results because there are more 6's in the workings box (SA)	Can you mend the dice gadget?
6.9.4 to 6.9.8	*Activity: R edits the workings to contain just one 6. They repeat 1000 trials. The pie chart shows even sectors .... I ask them to make a dice which favours 1's and 2's. They edit the workings to read:* ***choose-from [1 1 2 2 3 4 5 6]*** *and repeat 100 0 new trials. The pie chart shows most 1's and 2's and least 3, 4, 5, and 6's. I ask them to make the 1's more likely than the 2's. They edit the workings to read:* ***choose-from [1 1 1 2 2 3 4 5 6]****.*	

6.9.6 to 6.9.8	A & R : Higher numbers of trials generate even pie charts when the workings are fair (SA) A & R : Smaller numbers of trials result in the pie chart being less even.(SA)	

This mapping shows us how Rebecca's intuitions were re-shaped through the forging of connections between her intuitions of the cause and effect behaviour of deterministic phenomena with the evidence which she accrues from her interactions with the gadgets. It is not perhaps surprising that Rebecca's guiding intuitions were based on cause and effect. We had seen in the interviews how much she stressed unsteerability as an aspect of random behaviour. These interactions are guided by the forming of conjectures based on her intuitions at that time and from my own interventions where necessary.

Rebecca uses her intuitions of the deterministic when she can not relate the behaviour of the computer gadget with her perceptions of the everyday equivalent. Thus, when 100 trials first produced more tails and then more heads, she explains that real coins are influenced by how you flick it. My interventions lead Rebecca to trying 200 and then 1000 trials, as a result of which Rebecca concludes that the higher the number of trials, the more even is the pie chart.

This abstraction is though confined to the coin gadget and everyday coins. She does not automatically *transfer* this notion to the spinner gadget. Instead she uses deterministic-type intuitions to account for discrepancies between the computer's spinner and everyday spinners.

When the spinner gives uneven pie charts for 50 trials, Rebecca responds by editing the workings in attempts to redress the imbalance. Each of these attempts fails to generate an even pie chart until Rebecca conjectures that it may be a good idea to try more trials. Rebecca explicitly refers to their experience with the coin gadget, indicating clearly that she is the first time considering the possibility that this abstraction may have wider applicability than previously considered.

Even at this stage, the abstraction is situated in the coin and spinner gadgets where the workings are fair. We see how when Rebecca see the bias towards 6's in the dice gadget, she suggests that more trials may redress this imbalance. Rebecca is prepared to extend this abstraction from coins and spinners to the dice gadget much more quickly than she had been prepared to extend the abstraction from coins to spinners. The fact that the intuitions had proved reliable for both the spinner and the coin allowed he rot apply it confidently in this new context.

This confidence is misplaced; the dice still produces too many 6's even when she carries out 1000 trials. Rebecca recognises that the workings are unfair and that this accounts for the unevenness of the pie chart. This conjecture is confirmed by editing the workings and trying again.

The meanings shaped by her previous recent experiences with the coin compete with meanings about the connection between the workings box and the behaviour of the gadget, which are deterministically based. The large number intuitions are resilient in the face of evidence from a wide range of experiments, covering coins, spinners and dice. The global meanings prove to be reliable predictors of long term behaviour whereas they prove inadequate when the number of trials is relatively small. The really powerful idea is that the deterministic global meaning, that the workings box should control the shape of the pie chart for large small numbers, proves to be perfectly valid when the number of trials is large. Rebecca's efforts in hindsight were directed at discriminating between large number and large small number behaviour.

Anne's experience is in some ways similar to that of Rebecca's, though the evidence from the computer-based tools interacted in her case with local meanings for the behaviour of stochastic phenomena. Whereas for Rebecca issues of control had led to a cause and effect intuition playing a large part in the development of global meanings, for Anne we see unpredictability playing a stronger role. In her interactions with the coin, Anne refers to how the results can not be estimated. She is quick therefore to recognise that there does seem to be a pattern when larger numbers are used. Anne is tuned into predictability issues and so when she sees the pie chart appearing to be even she latches onto this pattern more quickly than does Rebecca.

Nevertheless, she does not easily accept that this abstraction extends to the spinner gadget, and it is Rebecca who first suggests that they might use more trials for the spinner. Once her intuition of predictability for large trials proves to be extensible to the spinner gadget, she assumes that this will generate an even pie chart for the dice. When the evidence suggests that the dice generated an uneven distribution of results even when the number of trials is large, she recognises that her predictability intuition is less reliable than expected. By editing the workings to make the dice fair and testing this new form of the dice gadget, Anne discriminates the limitations of that abstraction.

Compound events

From their profiles, we see that neither Rebecca nor Anne have strong global meanings. Both believe that each total for two dice is equally likely. Their reasoning is however different. Anne intuits that the totals must be equally likely because the dice are (individually) fair and so the combination of them must be fair. For Anne, fairness implies equiprobability. Rebecca also see the total of two dice as equiprobable because of a notion of fairness. For her though, the fairness arises out of consideration of control. Since the throws of the dice can not be controlled, they must be random, and she equates randomness to fairness.

In comparing the likelihood of doubles to specific totals, Anne and Rebecca differ. Anne sees throwing any doubles as harder than throwing a specific total such as 3, 5 or 7, because she actually replaces the specific total with the more general case of non-doubles. In contrast, Rebecca argues that the doubles are just the same as the specific totals because the unsteerability in both cases means that the events are random, therefore fair, and so equally likely.

These intuitions are confirmed early in their interactions with the two-spinner gadget (6.10.2) when I ask, "If these were real spinners, and we can get any total between 2 and 6, do you think there is any total is harder to get than the others; any total that's easier to get than any others?" Anne replies, "What it was in real life? *(I confirm.)* No." Rebecca adds, "There's a 50 / 50 chance of getting any total." I clarify, "So you think all the totals are equally easy or hard to get." They both say, "Yes."

The subsequent shaping of these intuitions through their interactions with this gadget and the two-dice gadget are summarised below.

Summary of paragraphs 6.10.3 - 6.11.9

When they inspect the pie chart for the results generated during their familiarisation period with the two-spinner gadget, there are more 4's and 3's than other totals (6.10.3). Rebecca says, "Maybe if we do it more times, it might be more even. It was with the dice and the spinner."

Rebecca conjectures that her situated abstraction may extend to the two-spinner gadget. Certainly she is not prepared to accept the evidence of just 50 trials

They carry out 1000 trials. Whilst we wait for the pie chart to appear, it becomes clear that both girls believe that the 3's and 4's will in fact appear more often. Rebecca explains, "Because most of the sums seem to come to either 3 or 4 .... The workings .... The second one down comes to 3 and then the third one down comes to 3. Then the next two down both come to 4."

Rebecca is attending to the workings and using her abstraction that the pie chart will not contain equal sectors even when the number of trials is large if the workings are not fair. Though she is now relating the likelihood of each total to the ways in which those totals can be broken down, we do not know that she thinks this is how everyday spinners would behave.

The pictogram shows that the total of 5 has not appeared (6.10.4). Anne suggests that there are no numbers for which you can make 5, but Rebecca points out that you can have 2 and 3 but this is missing from the workings. They edit the workings to read **choose-from [1+1 1+2 2+1 2+2 3+1 2+3 3+2 3+3]**. Rebecca is clear that 2+3 is needed as well as 3+2, "because the first number is representing the first spinner, so you have to have it both ways." She is equally clear that this is not necessary for doubles. They miss the fact that 1+3 is missing.

They repeat 1000 new trials with these workings (6.10.5). Anne predicts that the pie chart will be even, (presumably, from of her previous explanations, on the basis that the spinners are fair). The pie chart shows unequal sectors with least 2's. Anne says, "Maybe we should put 1 add 1 in again." Rebecca objects, "No, because it's already been done so it would just be the same." Anne argues, "Yes, because some of them have got the same again." Rebecca agrees "Yes, maybe we should actually put some of them in again, because then there's more chance of them coming out more even."

Anne see the insertion of another 1+1 as a way of equalising the sectors of the pie chart. Rebecca has a conflict between her intuition that the different totals should be equally likely (because they can not be steered) and her analysis that the doubles, unlike the non-doubles, do not need to be repeated in the workings.

I decide to intervene to offer support to Rebecca's view. I ask Anne what she means when she says that some of them are the same. Anne: "Well, 1 and 2 and 2 and 1 are the same .... they come to the same number." I say, "They come to the same total, but are they the same as far as the spinners are concerned?" Rebecca explains, "No they are not. Because, the second one down, that number *(pointing to the 1 of 1+2 in the workings box)* refers to that spinner *(pointing to the first* *spinner)*, and that number *(pointing to the 2 of 1+2)* refers to that spinner *(pointing to the second spinner)*. So, say, if that one *(the first spinner)* lands on 1 and that one *(the second spinner)* lands on 2, it would be three. And if that one *(the second spinner)* lands on 1 and that one *(the first spinner)* lands on 2, it would be three as well." Anne says, "Exactly .... I think we should add that one *(pointing to the 1+1)* and that one *(pointing to the 2+2)* again because then we get more of a chance of getting them." I say, "But then you would be putting in 1 plus 1 in twice." Anne responds, "Yes, because 2 doesn't come up as much, does it?" Rebecca agrees, "So maybe if we do that."

My intervention is rejected by Anne, which emphasises just how strongly she must feel that the two-spinners should generate equally likely totals. Under pressure from Anne, Rebecca allows her fairness intuition to dominate her analysis.

They begin to edit the workings (6.10.6). At one point Rebecca is about to add 2+2 as well as 1+1, but Anne points out that that would give 4 and they already have lots of 4's. Finally, the workings read: **choose-from [1+1 1+1 1+2 2+1 2+2 3+1 2+3 3+2 3+3 3+3]**. The inconsistency of adding an extra 1+1 and not an extra 2+2 to the workings is not a concern for Anne since her aim is simply to equalise the likelihoods of the different sectors, and hence make the gadget work properly in her view.

After 1000 new trials, the pie chart is fairly even (6.10.7). I wish to draw their attention to the inconsistency of their solution. I point out, "What we don't know for sure is whether that is how real spinners would behave. I think what you need to try and do is justify why you should have 1 plus 1 in there twice over." Rebecca replies, "Because everything else has two ways of coming except maybe 2 plus 2." I ask, "But in reality, does 1 plus 1 have two different ways of coming?" Rebecca argues, "I think it is more fair because the pie chart looks roughly even and before there were barely any 2 and barely any 4 ..." I respond, "I think you have certainly made it more fair. What I am not convinced about is that you have made it more like real spinners would be .... maybe with real spinners that would not be the case." Anne: "Oh, yes, mmm." I say (6.10.8), "You see, I am not sure you are being fair by putting 1+1 in twice." She continues, "We don't want it to be even. We want it to work like a real spinner."

This heavy intervention has raised the possibility that the computer's gadget may be fair but that type of fairness may be unrealistic. In a sense, I argue, they are being unfair by not dealing with 1+1 and 2+2 in similar fashion.

They edit the workings to read: **choose-from [1+1 1+2 2+1 2+2 3+1 2+3 3+2 3+3]**. I point out the missing 1+3, which they then insert. They generate 1000 new trials and I ask what the pie chart will look like. Anne replies, "A bit uneven .... because 1 and 1 has only got once, because that is what a real spinner would be like. And the rest has got like double number and it can make different numbers. " Rebecca: "I think maybe .... 2 won't come up as much and 6." The pie chart shows most 4's and least 2's and 6's, with slightly less 6's than 2's.

I ask why the 4 should be bigger (6.10.9). They look at the workings. Rebecca says, "I've seen it twice .... It's written there three times." They then continue by comparing how many times each of the other totals is represented in the workings. I ask, "Now, how do you think this compares to doing it with two real spinners?" Anne replies, "Probably it would be about the same because we are trying to work it as a real spinner, and we've got the same sort of numbers." Rebecca agrees. I ask, "So do you think the different totals, 2, 3, 4, 5, 6 are all just as easy to get or all just as hard to get. Or is there one that's easy to get?" They both say 4 is easier to get. And I ask if there is one that is hard. They both say, "6". I ask, "Just the 6?" They then say, "2". Rebecca adds, "But 6 seems harder because it is smaller ...." Anne interjects, "Yes, much smaller." Rebecca continues, "Less of 6."

Rebecca and Anne are now confident that everyday spinners will behave in this way. With my guidance, and the evidence from interacting with the two-spinners gadget, they have abstracted the notion that the number of ways that the total can be represented is more important than notions of fairness. It is interesting however that this abstraction is not so robust and can be dominated by other factors. Hence, both girls think the 6 is less likely than the 2 because the results seem to point in that direction, even though the workings suggest that they are equally likely.

I suggest that they repeat the experiment again (6.10.10). This time the pie chart shows that 6's occur less often than the 2's. Rebecca and Anne agree that the 6 and the 2 are in fact equally hard to get.

Anne and Rebecca begin to use the two-dice gadget. It is clear from their early discussions that they believe there are missing data in the workings (6.11.1). In particular, Rebecca notices, "There's lots of different ways .... There's probably more ways of making 6 .... " Anne points out, "Ah, 5 add 1 is missing to make 6." They insert 5+1 below 1+1 in the workings.

I ask about two real dice (6.11.2), ""If we were shaking two real dice, do you think all the totals you could get are just as easy, just as hard, or do you think some totals are easier than others, harder than others?" "50 / 50 chance of getting them." Anne agrees. I clarify, "So you think they are all about the same chance?" They both say, "Yes."

Anne and Rebecca recognise that there are data missing from the workings and therefore the gadget may not yet work like everyday dice, which, being unsteerable in Rebecca's view, would have equally likely totals.

At this point, they begin to notice some other missing possible outcomes (6.11.3). After several additions, the workings read: **choose-from [1+1 5+4 4+5 2+3 5+1 1+2 2+1 2+2 3+3 4+4 5+5 6+6 3+2]**. Rebecca suggests that they repeat it a number of times (6.11.4) to get a clue which ones are missing. They repeat 1000 new trials. I ask, "What do you think it *(referring to the pie chart)* should look like if it were two real dice?" They both say that it will be fairly even. The pie chart is uneven and shows most 6's and least 2's.

This episode clarifies that Rebecca believes that the two dice will not generate a pie chart with equally sectors because they have not yet found all the missing data for the workings.

The pictogram shows that there are no 7's and 11's (6.11.5). They insert 6+5 and 5+6 at the bottom of the workings box. They then append 5+2, 2+5, 3+4, 4+3, 6+1 and 1+6. I suggest that they check the other total systematically. They work through totals 2 through to 12. Eventually they have all 36 combinations.

They repeat 1000 new trials (6.11.6). I ask what they think the pie chart will look like. Anne replies, "Fairly even? ....Some of the numbers might not be because there's not as much as the other number." Rebecca says, "Maybe roughly even because now that we have got all the sums. I'm not too sure at the moment." Anne adds, "I think some will be a bit less because they haven't got as much as the others .... because some of the numbers will not be the same, will be less, because we didn't find enough sums for them .... like 1 add 1." I ask, "Can you give me an example of one that had a lot of different ways of getting it." Anne: "7."

This conversation shows that Anne and Rebecca are beginning to question their earlier assumption. Although they still think the pie chart may be even, they have noticed that some totals are more represented than others in the workings. This fact has been brought to their attention by the process of editing the workings themselves. Their purpose for inserting the extra data had been to equalise the sectors in the pie chart, making the two-dice gadget consistent with their intuitions about the totals of everyday dice. This editing process alerted them to the unequal representation of different totals. They had their previous experience of the two-spinners gadget to call upon but there is not explicit evidence that they were doing this. We might tentatively infer that their lack of certainty points to a competition between two elements of their construction of meaning:

• the abstraction situated in two-spinner domain that the frequency of the totals reflects how many ways that total is made in the workings,

• the intuition that the dice are unsteerable and fair and so the totals are equally likely.

The pie chart shows most 7's and least 2's and 12's (6.11.7). Rebecca's first reaction is that they have not got them all. I reassure her that we have. Rebecca says, "Ah, I bet there are various ways of making a number. There can be more ways of making one number than there can be of another." They identify that 12's, 2's and 3's do not have many ways whereas 7 has the most.

We return to the pie chart and I ask why it looks like that (6.11.8). Anne says, "Because some of the sums we put as more. Like 2, we could only get about one of them so that's why it came out a bit like that."

I wonder whether they will be prepared to extend the domain of this abstraction to everyday contexts. I ask, "So when you home, and you are playing dice with your brother, Rebecca, and you've got two dice, and you say to him, 'I bet you 10p what the total's going to be.' You go for 7. What are you going to tell him to go for?" Rebecca: "2 .... no, sorry 12 .... 12 is smaller than 2." I ask, "So, do you now think that all the totals on two dice are just as easy or just as hard as each other? Or do you think some totals are easier than others?" They both say that some totals are easier to get.

Rebecca extends her abstraction that the totals are not equally likely and that how likely they are depends upon how often they are represented in the workings. This abstraction is as true of two dice as it was of two spinners and extends beyond the Chance-Maker microworld to everyday contexts. Again, as with two-spinners, where the least likely totals are equally represented, she gives the casting vote to the actual frequencies in the results.

Indeed, Rebecca does not see the totals of two-dice as random any more (6.11.9). Rebecca says, "It doesn't seem random now, does it? ....Well, not really random because they are not 50 / 50 .... It's not 50 / 50 because you don't get the same number."

We can track the development of Anne and Rebecca's intuitions as they interact with the external resources in the form of computer-based tools and my interventions.

Para	Conjectures and Situated Abstractions	Critical Interventions
6.10.2 to 6.10.3	Two spinners in everyday contexts are fair and so the totals are equally likely (SA) The size of the sectors in the pie chart reflect the number of ways the totals are represented in the workings provided the number of trials is large (SA)	
6.10.4 to 6.10.7	*Activity: A & R do 1000 new trials. The pictogram shows that 5 is missing. They edit the workings to include 2+3 and 3+2. They repeat a new 1000 trials. The pie chart is uneven. A & R argue about whether the doubles should be written twice into the workings. Anne's argument that putting 1+1 in again, but not 2+2, would equalise the pie chart. persuades Rebecca.. They edit the workings to include another 1+1. They repeat 1000 new trials. The pie chart looks even.*	
6.10.7		You have made the gadget fair but real spinners may not be fair. You have put 1+1 in twice but 2+2 only once. Is it fair to put 1+1 into the workings twice?
6.10.7 to 6.10.8	*Activity : A & R edit the workings to exclude one of the 1+1's. I point out that 1+3 is also missing and this is entered. They repeat 1000 new trials and the pie chart shows most 4's and least 2's and 6's.*	
6.10.9 to 6.10.10	4 is easier to get than the other totals because there are more ways of making 4 (SA) 6 and 2 are harder to get because they have less ways in the workings, but 6 is even harder because it appeared less times in the results (SA)	Repeat the experiment to see if you get the same again.

6.10.10	*Activity: A & R repeat 1000 new trials. The pie chart is similar except that the 6's are more frequent than the 2's.*	
6.10.11	6 is the same as 2 (SA) 4 is easier to get than 2 for everyday spinners as well as computer gadgets (SA)	
6.11.1	*Activity: A & R begin to use the two-dice gadget. They think that some totals are missing and being to insert data into the workings..*	
6.11.2	A & R: Totals for two everyday dice are equally hard or easy (SA)	
6.11.3 to 6.11.6	*Activity: A & R make further additions to the workings and decide to repeat 1000 new trials to help identify which other ones are missing. The pictogram shows 7's and 11's are missing. They add combinations that make 11 and 7. They then continue systematically until they have all 36 combinations..*	
6.11.6	Perhaps the pie chart will not be even because some totals appear more often in the workings (C)	
6.11.6 to 6.11.7	*Activity: A & R repeat 1000 new trials.. The pie chart shows most 7's and least 2's and 12's.*	
6.11.8	Some totals are more likely than others, even for everyday dice (SA) R: Two dice are not really random because the totals are not 50/50. (SA)	

The above chart shows how the evidence from their interactions with the computer's resources raised competing intuitions and situated abstractions. They try to construct meaning for the behaviour of the two-spinners and two-dice gadgets. Their activities give us a window into those intuitions.

We see how the girls try to manipulate the workings to make the gadgets behave fairly. This is such an important goal for them that they are prepared to represent the different combinations in the workings unfairly and inconsistently. At this stage my interventions were necessarily strong since it seems unlikely that, without those interventions, they would have been content that the pie chart was showing the evenness that they had expected.

The possibility that the two spinners may not be fair in everyday contexts has been opened up and they gather more evidence from the computer. The now abstract the notion that the sectors in the pie chart are proportional to the number of ways that the total can be made in the workings.

This abstraction is not automatically *transferred* to the two-dice gadget. It requires a considerable amount of interaction with this gadget before their initial intuition that all totals are equally likely is discarded in favour of an extension of the abstraction for two-spinners to the two-dice context. Crucially it is their own editing of the workings box which points up that some totals appear more often than others and it is this process that helps them to make the connection between this gadget and the two-spinners gadget.

NOTE : THE OTHER CASE ANALYSES CAN BE FOUND ON THE WORLD WIDE WEB AT ADDRESS: http://www.warwick.ac.uk/wie/staff/DP.htm

A4. CONTENTS OF APPENDICES ON WORLD WIDE WEB

These appendices can be accessed on the World Wide Web at address:
http://www.warwick.ac.uk/wie/staff/DP.htm

* = *Also included in appendices of main thesis.*